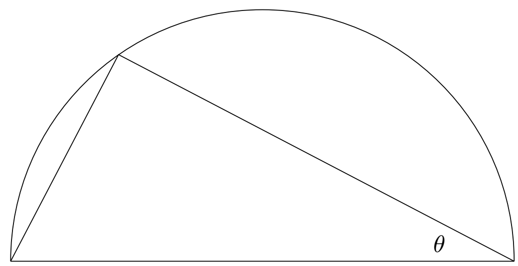
March 2, 2015

1. A triangle is inscribed in a semi-circle of radius r as shown in the figure:



The area of the triangle is

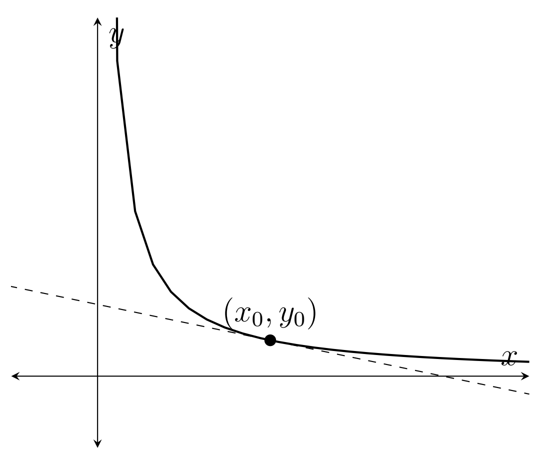
(A) $r^2 \sin 2\theta$ (B) $\pi r^2 - \sin\theta$ (C) $r \sin\theta \cos\theta$ (D) $\pi r^2/4$ (E) $\pi r^2/2$

Solution: (A)

The triangle is right with hypotenuse a diameter of the circle, hence of length $2r$. The sides a and b of the triangle have lengths $2r\cos\theta$ and $2r\sin\theta$, hence the area of the triangle is

$$A = \frac{1}{2}ab = \frac{1}{2}(2r\cos\theta)(2r\sin\theta) = r^2 \sin 2\theta.$$

2. A triangle is formed by the coordinate axes and the tangent line at the point with coordinates (x_0, y_0) lying on the curve $\mathscr{C} = \{(x, y) : xy = k\}$, as shown in the figure.



The numbers x_0, y_0, and the slope m of the tangent line satisfy $y_0 = -mx_0$. The area of the triangle is

(A) $2k$ (B) $2k^2$ (C) $x_0 + 2k$ (D) $k(x_0 + y_0)$ (E) $k/(x_0 + y_0)$

Solution: (A)

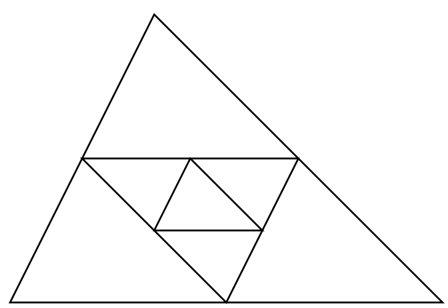
The point-slope form of the equation of the line is

$$y - \frac{k}{x_0} = \left(-\frac{y_0}{x_0}\right)(x - x_0). \tag{1}$$

The triangle is a right triangle with height the value of y when we set $x = 0$ in (1), $H = 2k/x_0$, and base the value of x when we set $y = 0$ in (1), $B = 2x_0$. Thus the area is

$$A = \tfrac{1}{2}HB = 2k.$$

3. There are three triangles of different sizes: small, medium and large. The small one is inscribed in the medium one such that its vertices are at the midpoints of the three edges of the medium one. The medium triangle is inscribed in the large triangle in the same way as shown in the figure. If the small triangle has area 1, what is the sum of the areas of the three triangles?



(A) 14 (B) 16 (C) 19 (D) 21 (E) 25

Solution: (D)

The larger triangle has area 4 times that of the inscribed one. So the medium triangle has area 4 and the large triangle has area 16 and the total area is $1 + 4 + 16 = 21$.

4. Describe the shape of the graph of $f(x) = \dfrac{6x^2 + 7x - 3}{2x + 3}$.

(A) hyperbola (B) parabola (C) circle (D) full line (E) line with a hole

Solution: (E)

$6x^2 + 7x - 3$ factors as $6x^2 + 7x - 3 = (3x - 1)(2x + 3)$, hence, for $x \neq -3/2$ the function is given by $f(x) = 3x - 1$. This is the equation of a line. The number $-3/2$ is not in the domain, there is a hole on the line at $(-3/2, -11/2)$.

5. How many of elements of the set $\{-12.3, -9, -5, 1, 2.14, 3.6, 5.2, 7.8, 101\}$ are **not** solutions of the equation $|2x^2 - 9x + 6| = 2x^2 - 9x + 6$?

(A) 2 (B) 3 (C) 5 (D) 6 (E) 8

Solution: (B)

The equation is equivalent to the inequality $2x^2 - 9x + 6 \geq 0$, whose solution is given by $x \leq (9 - \sqrt{33})/4 \approx 0.8138\ldots$ or $x \geq (9 + \sqrt{33})/4 \approx 3.6861\ldots$

6. The number of elements in the intersection of the solution set of $\dfrac{\sqrt{2}\cdot x - 1}{2x^2 - 4\sqrt{2}\cdot x + 4} > 0$ with the set $\{-7, -5, -2, -1, 1, \sqrt{2}, \sqrt{107}, \sqrt{108}\}$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: (D)

The denominator is $2(x - \sqrt{2})^2$, which vanishes at $x = \sqrt{2}$, and it is positive for all other values of x. Hence the inequality is equivalent to $\sqrt{2}\cdot x - 1 > 0$ but $x \neq \sqrt{2}$, that is, $x > 1/\sqrt{2}$ but $x \neq \sqrt{2}$.

7. How many positive integers are in the range of the function $f(x) = -2x^2 + 8x - 3$?

(A) 0 (B) 2 (C) 3 (D) 5 (E) 6

Solution: (D)

Since $f(x) = -2(x-2)^2 + 5$, the maximum value of f is 5, the function is continuous and not bounded from below.

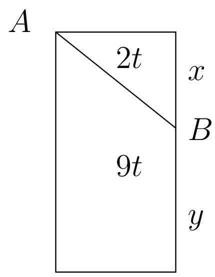
8. The number of elements in the intersection of $A = \left\{-1, -\dfrac{1}{2}, 0, \dfrac{1}{2}, 1, \dfrac{3}{2}, 2\right\}$ with the solution set of $\left(1 + \dfrac{x+2}{x-1}\right)\left(1 + \dfrac{x-3}{2}\right) > 0$ is

(A) 01 (B) 1 (C) 2 (D) 3 (E) 4

Solution: (E)

The left hand side simplifies to $\dfrac{2x+1}{x-1} \cdot \dfrac{x-1}{2} > 0$ which is meaningless for $x = 1$ and otherwise it is $\dfrac{2x+1}{2} > 0$. Thus the solution of the inequality is $x > -\dfrac{1}{2}$ but $x \neq 1$.

9. The line segment AB divided the rectangle in the picture into two parts. The proportion of the areas of these parts is $2 : 9$.



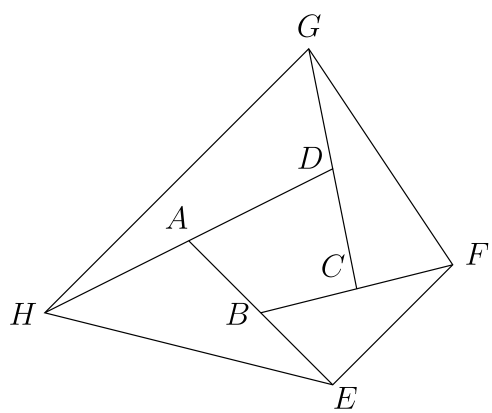
In what proportion are the lengths x and y?

(A) 2 : 9 (B) 2 : 7 (C) 4 : 7 (D) 4 : 9 (E) 5 : 9

Solution: (C)

Introducing w for the width of the rectangle and using the notation in the figure, we get $2t = x \cdot w/2$ and $9t = (x + y + y) \cdot w/2$. Dividing the second equation by the first yields $9/2 = (x + 2y)/x$. Thus $7/2 = 2y/x$ and $y/x = 7/4$.

10. Consider a convex 4-gon $ABCD$. Reflect A about B to get E, B about C to get F, C about D to get G and D about A to get H. What percentage is the area of $ABCD$ of the resulting 4-gon $EFGH$?



(A) 10% (B) 20% (C) 25% (D) 30% (E) 33.5%

Solution: (B)

For example, the area of the triangle BEF is twice the area of the triangle ABC since AB and BE have the same length, but the distance of F from the line BE is twice the distance of C from the line AB. Similarly the area of the triangle DGH is twice the area of ACD, and so on. Hence the combined area of the triangles BEF and DGH is twice the area of $ABCD$. Similarly the combined area of the triangles CFG and AHE is also twice the area of $ABCD$.

11. What is the number of real solutions of the equation $2015x^6 + 4030x^3 + x^2 + 2x + 2016 = 0$ if we count each different solution only once, **without multiplicities**?

    (A) 0 (B) 1 (C) 2 (D) 4 (E) 6

    Solution: (B)

    The equation may be rewritten as $2015(x^6 + 2x^3 + 1) + x^2 + 2x + 1 = 0$, that is, $2015(x^3 + 1)^2 + (x + 1)^2 = 0$. The left hand side is the sum of two squares of real numbers, which can be zero only if neither of these squares is positive. The only solution is $x = -1$. (Note that $x^3 + 1 = (x + 1)(x^2 - x + 1)$ so $(x + 1)^2$ is a factor of the left hand side, but $(x + 1)^3$ is not. The multiplicity of $x = -1$ is 2.)

12. How many four-digit positive integers are there with exactly two even and two odd, pairwise different digits?

    (A) 2008 (B) 2100 (C) 2160 (D) 2240 (E) 2640

    Solution: (C)

    Let us denote the position of the even digits by e, and the position of the odd digits by o. There are 6 possible patterns: *eeoo*, *eoeo*, *eooe*, *oeeo*, *oeoe* and *ooee*. If the first digit is even, we may choose it 4 ways (we can not begin with 0), and the other even digit may be any of the hitherto unused 4 digits. Thus, in this case, we may choose the even digits $4 \cdot 4 = 16$ ways. If the first digit is not even, the even digits may be selected $5 \cdot 4 = 20$ ways. In either case, the odd digits may be selected $5 \cdot 4 = 20$ ways. Thus the number of positive integers satisfying all our criteria is $3 \cdot 16 \cdot 20 + 3 \cdot 20 \cdot 20 = 2160$.

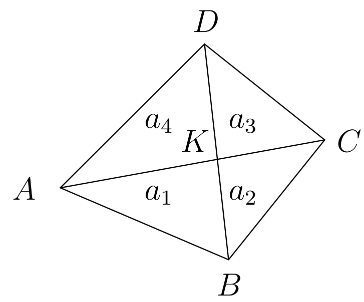
13. What is the nth digit after the decimal point in the decimal representation of $(\sqrt{26} - 5)^n$.

    (A) 0 (B) 1 (C) 6 (D) 5 (E) Can not be determined

    Solution: (A)

    We have $\sqrt{26} - 5 = \frac{26 - 25}{\sqrt{26} + 5} = \frac{1}{\sqrt{26} + 5} < \frac{1}{5 + 5} = 10^{-1}$. That is, we have $0 < \sqrt{26} - 5 < 10^{-1}$. Thus $0 < (\sqrt{26} - 5)^n < 10^{-n}$. Therefore, in the decimal expansion of $(\sqrt{26} - 5)^n$ the nth digit after the decimal point must be a zero.

14. The diagonals of the convex four-gon $ABCD$ divide the four-gon into four triangles, whose areas are a_1, a_2, a_3, and a_4 respectively, as shown in the figure.



Which of the following identities must hold for these areas?

(A) $a_1 + a_2 = a_3 + a_4$ (B) $a_1 + a_3 = a_2 + a_4$ (C) $a_1 \cdot a_2 = a_3 \cdot a_4$ (D) $a_1 \cdot a_3 = a_2 \cdot a_4$
(E) $a_1 - a_3 = a_2 - a_4$

Solution: (D)

Let d_A, respectively d_C denote the distance of A, respectively C from the line BD. Then $\frac{a_1}{a_2} = \frac{BK \cdot d_A/2}{BK \cdot d_C/2} = \frac{d_A}{d_C}$ and $\frac{a_4}{a_3} = \frac{DK \cdot d_A/2}{DK \cdot d_C/2} = \frac{d_A}{d_C}$. Thus $a_1/a_2 = a_4/a_3$ and so $a_1 \cdot a_3 = a_2 \cdot a_4$.

15. The lengths of 3 edges, meeting in one vertex, of a rectangular box, measured in centimeters are distinct positive integers. One of the edges has length 6 cm. What is the sum of the lengths of the other two edges if the number expressing the volume of the box in cubic centimeters is the same as the number expressing its surface area in square centimeters.

(A) 10 (B) 12 (C) 14 (D) 16 (E) 20

Solution: (D)

Let us denote the lengths of the edges by a, b and c, assuming $a = 6$. The equality between the volume and the surface area may be written as $2(6b + 6c + bc) = 6bc$ which may be simplified to $3(b + c) = bc$. Since b and c are integers, the left hand side is a multiple of 3, and so either b or c on the right hand side is also a multiple of 3. Without loss of generality we may assume $b = 3b_0$ for some positive integer b_0. Thus we get $3b_0 + c = b_0c$ or $3b_0 = (b_0 - 1)c$. If $b_0 = 1$ then the right hand side is zero, yielding $b_0 = 0$, a contradiction. Thus b_0 is at least 2 and $b_0 - 1 \geq 1$ divides $3b_0$. Since $b_0 - 1$ and b_0 are relative primes, $b_0 - 1$ must divide 3, so $b_0 - 1 \in \{1, 3\}$ and $b_0 \in \{2, 4\}$. Setting $b_0 = 2$ yields $3 \cdot 2 = (2 - 1)c$ and $c = 6$, in contradiction with the edges being pairwise distinct. Thus we can only have $b_0 = 4$, $b = 3b_0 = 12$ and $c = 3b_0/(b_0 - 1) = 4$.

16. The National Assembly of a country has 10 committees. Every member of the National Assembly works in exactly two committees and any pair of committees has exactly one member in common. How many members does the National Assembly have?

(A) 25 (B) 30 (C) 45 (D) 50 (E) 60

Solution: (C)

Let us represent each member and each committee with a vertex. Connect each vertex representing a member to each vertex representing a committee, of which (s)he is a member of. We obtain a graph, which may also be created as follows. Take 10 vertices, representing the 10 committees, connect each pair of these committees by an edge. Subdivide each edge connecting two committees by inserting the midpoint of the edge and labeling it with the only member of the National Assembly that belongs to both committees. There are $10 \cdot 9/2 = 45$ pairs of committees, so we must have 45 members in the National Assembly.

17. What is the sum of all integers n, for which $Q(n) = \dfrac{4n^2 - 4n - 24}{n^3 - 3n^2 - 4n + 12}$ is also an integer?

(A) 10 (B) 11 (C) 12 (D) 13 (E) infinity

Solution: (B)

$Q(n) = \dfrac{4(n+2)(n-3)}{(n-2)(n+2)(n-3)}$ is undefined for $n \in \{-2, 2, 3\}$ for all other values of n we have $Q(n) = \frac{4}{n-2}$. We are looking thus we are looking for all values of n such that $n - 2$ divides 4 but $n \notin \{-2, 2, 3\}$. The set of divisors of 4 is $\{\pm 1, \pm 2, \pm 4\}$ so the set of possible values of n is $\{1, 3, 0, 4, -2, 6\} \setminus \{-2, 2, 3\}$, that is, $\{1, 0, 4, 6\}$. The sum of these elements is 11.

18. We draw all diagonals of a convex 12-gon. Each intersection of diagonals is contained in exactly two diagonals. How many intersections of diagonals are there inside the 12-gon?

(A) 216 (B) 480 (C) 495 (D) 500 (E) 512

Solution: (C)

If the intersection of the two diagonals is inside the 12-gon, then the endpoints of these diagonals form a convex 4-gon, whose diagonals are the selected two diagonals. The converse is also true. There are $\binom{12}{4} = 495$ ways to select 4 vertices out of 12.

19. What is the **exact** value of $\dfrac{\sin(3\alpha) + \sin\alpha}{\sin(2\alpha)\cos(\alpha)}$ (for all values of α such that the above expression is defined)?

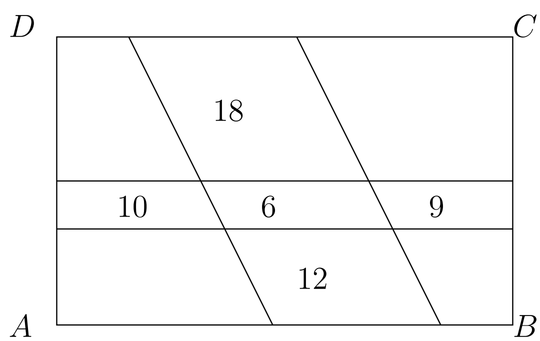
(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Solution: (E)

We have

$$\begin{aligned} \frac{\sin(3\alpha)+\sin\alpha}{\sin(2\alpha)\cos(\alpha)} &= \frac{\sin(2\alpha+\alpha)+\sin(2\alpha-\alpha)}{\sin(2\alpha)\cos(\alpha)} \\ &= \frac{\sin(2\alpha)\cos(\alpha)+\cos(2\alpha)\sin(\alpha)+\sin(2\alpha)\cos(\alpha)-\cos(2\alpha)\sin(\alpha)}{\sin(2\alpha)\cos(\alpha)} \\ &= \frac{2\sin(2\alpha)\cos(\alpha)}{\sin(2\alpha)\cos(\alpha)} = 2. \end{aligned}$$

20. We subdivide the rectangle $ABCD$ with two pairs of parallel lines into 9 parts, as shown in the figure. The areas of five of these parts, measured in square centimeters is provided. What is the area of the rectangle $ABCD$, measured in square centimeters?



(A) 150 (B) 190 (C) 180 (D) 155 (E) 200

Solution: (A)

Let a be the length of AB and let b be the length of BC. Let x be the length of the side of the parallelogram of area 12 that is parallel to AB. The heights of the parallelograms are then $12/x$, $6/x$ and $18/x$. So we have $b = 12/x + 6/x + 18/x = 36/x$. The area of the central horizontal stripe is $a \cdot 6/x = 10 + 6 + 9$, so $a = 25x/6$. The area of the rectangle $ABCD$ is $ab = 25x/6 \cdot 36/x = 150$.

21. What is the sum of the squares of the solutions of the equation $\log_3(x) + \log_x(9) = 3$?

(A) 0 (B) 9 (C) 12 (D) 81 (E) 90

Solution: (E)

The equation is only defined if $x > 0$ and $x \neq 1$. Using the change of base formula we get $\log_3(x) + \dfrac{\log_3(9)}{\log_3(x)} = 3$. Introducing $u = \log_3(x)$ we may write $u + \dfrac{2}{u} = 3$. This implies $u^2 - 3u + 2 = 0$, hence $u = 1$ or $u = 2$. Thus $x = 3^u$ is either 3 or 9. The sum of the squares of the solutions is $3^2 + 9^2 = 90$.

22. In the language BadSpeak, the alphabet contains only the letters a, b, and c. How many 4-letter words in BadSpeak contain the letter "c"?

    (A) 81 (B) 108 (C) 108 (D) 65 (E) 1

    Solution: (D)

    There are $3^4 = 81$ 4-letter words in BadSpeak. Of these, $2^4 = 16$ do not contain the letter "c." Hence $81 - 16 = 65$ words do contain the letter "c."

23. If $\log x = -8$ and $\log y = 14$, then $\log x^2y^3 =$?

    (A) -112 (B) 26 (C) 12544 (D) 6 (E) 0

    Solution: (B)

    $\log x^2y^3 = 2\log x + 3\log y = 2(-8) + 3(14) = 26.$

24. Let w be a real number. What is the sum of the (possibly complex) roots of the equation $x^2 + 13x + w = 0$?

    (A) w (B) $-w$ (C) $13 + w$ (D) $-13 - w$ (E) -13

    Solution: (E)

    Let the roots be c, d. Then $x^2 + 13x + w = (x - c)(x - d) = x^2 - (c + d)x + cd$. Hence the sum of the roots is $c + d = -13$.

25. An $a \times b \times c$ block of unit cubes has faces with surface areas $48, 60,$ and 80. Suppose the six faces are painted, after which the block is cut into unit cubes. Which of the following is the probability that a randomly selected cube is painted on exactly two faces?

    (A) 1/2 (B) 3/20 (C) 2/21 (D) 7/60 (E) 1/8

    Solution: (B)

    First we compute a, b, and c from $ab = 48$, $bc = 60$ and $ac = 80$. Dividing the product of the second two equations by the first gives $c^2 = 100$, that is, $c = 10$. Using this information we get $b = 6$ from the second equation and $a = 8$ from the first equation. There are $4(4 + 6 + 8) = 72$ unit cubes with paint on exactly 2 faces. The total number of all cubes is $6 \cdot 8 \cdot 10 = 480$. The probability is $72/480 = 3/20$.