March 2, 2015

1. If two solutions of $x^3 + px + q = 0$ are $x_1 = 1$ and $x_2 = 2$ then q equals

   (A) 3 (B) -3 (C) 6 (D) -6 (E) 2

   Solution: (C)

   Let us denote the third root by x_3. By the Viète formulas, the coefficient of x^2 is $-x_1 - x_2 - x_3$. This coefficient is zero, so we have $x_3 = -x_1 - x_2 = -3$. Also by the Viète formulas, we have $q = -x_1x_2x_3 = 6$.

2. How many four-digit integers are there with no two consecutive zeros?

   (A) 8919 (B) 8829 (C) 8001 (D) 4232 (E) 4156

   Solution: (B)

   The number of all four-digit positive integers is 9000, as the first digit can be chosen 9 ways, and all other digits can be chosen 10 ways. From this number we subtract the number of four digit integers containing at least two consecutive zeros. There are exactly 9 four-digit integers containing 3 consecutive zeros: these are the multiples of 1000, up to 9000. We are left to count all four digit integers that contain exactly one pair of consecutive zeros (and no other zero). These are either of the form $x00y$ or of the form $xy00$ where x and y are nonzero digits. There are $9 \times 9 = 81$ integers of each kind. Thus the number of all four-digit integers with no two consecutive zeros is $9000 - 2 \cdot 81 - 9 = 8829$.

3. A regular octahedral die has faces numbered 1 though 8. It is rolled four times obtaining a sum S satisfying $4 \leq S \leq 32$. What is the probability that $S(36 - S) < 180$?

   (A) 0 (B) 1/512 (C) 3/1024 (D) 5/2048 (E) 7/4096

   Solution: (D)

   The graph of $f(x) = x(36 - x) - 180$ is a concave down parabola, with x-intercepts $x = 6$ and $x = 30$. Thus an integer S, satisfying $4 \leq S \leq 32$, also satisfies $S(36 - S) < 180$ exactly when $S \in \{4, 5, 31, 32\}$ holds. $S = 4$ can be achieved in exactly one way, by rolling four 1's. $S = 5$ can be achieved in exactly four ways, by rolling three 1's and a 2. Similarly, $S = 32$ can be achieved only one way (four 8's) and $S = 31$ can be achieved in exactly four ways (three 8's and one 7). So the probability is $(4+1+4+1) \div 8^4 = 5/2048$.

4. What is the sum of the coordinates of the point (x, y) that satisfies both $x^2 + (y - 5)^2 = 4$ and $(x - 12)^2 + y^2 = 121$?

   (A) 75/13 (B) 77/13 (C) 79/13 (D) 81/13 (E) 83/13

Solution: (C)

There is only one point common to the circle of radius 2, centered at $(0,5)$, and to the circle of radius 11, centered at $(12,0)$. (Note that the distance of the two centers is $\sqrt{5^2+12^2}=13$ which is the sum of the radii.) The point lies along the line joining the centers of the two circles. It is given by $(x,y)=\frac{11}{13}(0,5)+\frac{2}{13}(12,0)=(\frac{24}{13},\frac{55}{13})$, so the answer we need is $\frac{55}{13}+\frac{24}{13}=\frac{79}{13}$.

5. A teacher can grade 12 tests per hour by herself. Working with an assistant, she can grade 20 tests per hour, but it takes two hours to train the assistant. Let N be the least number of tests that can be graded faster by training an assistant. Then N belongs to the following interval:

   (A) $60-79$ (B) $80-119$ (C) $120-239$ (D) $240-244$ (E) $245-299$

   Solution: (A)

   We solve $p/12=2+p/20$ for the number p of papers required to break even and get $p=60$. There needs to be at least 61 tests to make the training of the assistant worthwhile.

6. What is the greatest integer k such that 3^k divides 100!

   (A) 33 (B) 44 (C) 47 (D) 48 (E) 50

   Solution: (D)

   The number $\lfloor\frac{100}{3}\rfloor=33$ counts the number of multiples of 3 in the range 1 to 100. The number $\lfloor\frac{100}{9}\rfloor=11$ counts the number of multiples of 9 in the range 1 to 100. The number $\lfloor\frac{100}{27}\rfloor=3$ counts the number of multiples of 27 in the range 1 to 100. The number $\lfloor\frac{100}{81}\rfloor=1$ counts the number of multiples of 81 in the range 1 to 100. So the answer we seek is the sum $33+11+3+1=48$.

7. What is the remainder when the number $N=1^2+3^2+5^2+\cdots 99^2$ is divided by 100?

   (A) 0 (B) 50 (C) 60 (D) 70 (E) 80

   Solution: (B)

   The sum of the first n squares is given by $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$. Note that $\sum_{i=1}^{100} i^2=N+4\sum_{i=1}^{50} i^2$, it follows that $N=\frac{100(101)(201)}{6}-4\frac{50(51)(101)}{6}=\frac{101}{6}[100(201)-200(51)]=166650$, and we can see that this number the last two digits of this number are 50.

8. What is the largest value of n such that division of each of the numbers $1621, 2237$, and 2545 by n leave the same remainder?

(A) 7 (B) 11 (C) 33 (D) 77 (E) 308

Solution: (E)

n is the greatest common divisor of $2237 - 1621 = 616 = 2^3 \cdot 7 \cdot 11$ and of $2545 - 1621 = 924 = 2^2 \cdot 3 \cdot 7 \cdot 11$, that is, $2^2 \cdot 7 \cdot 11 = 308$.

9. For how many integers k is it true that $\frac{k^3+8}{k^2-4}$ is an integer?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: (D)

We can not divide by zero so $k \neq \pm 2$. For all other values of k, the expression is equivalent to $\frac{k^2-2k+4}{k-2} = k + \frac{4}{k-2}$, which is an integer exactly when $k - 2 \in \{\pm 1, \pm 2, \pm 4\}$, or equivalently, $k \in \{1, 3, 0, 4, -2, 6\}$. Excluding $k = -2$ leaves us with $k \in \{0, 1, 3, 4, 6\}$.

10. The curve $y = 9x^2 + 8x + |c|$ has at least one real zero for all c such that $a \leq c \leq b$. What is $b - a$?

(A) 16/9 (B) 24/9 (C) 32/9 (D) 49/9 (E) 51/9

Solution: (C)

Complete the square to get $9x^2 + 8x + \frac{16}{9} + |c| - \frac{16}{9} = 9(x + 4/9)^2 + |c| - \frac{16}{9}$, so $|c| - \frac{16}{9} \leq 0$, and it follows that c must satisfy $-\frac{16}{9} \leq c \leq \frac{16}{9}$.

11. The curve $(|y| - x^2)(|x| - y^2) = 0$ divides the plane into some bounded and some unbounded regions. How many such regions are there?

(A) 4 (B) 6 (C) 8 (D) 12 (E) 16

Solution: (D)

The curve is the union of four standard parabolas, all obtainable from $y = x^2$ by rotating it about the origin by multiples of 90 degrees. There are four bounded regions and 8 unbounded regions determined by the curve.

12. Find the last digit of $1 + 3 + 3^2 + \cdots + 3^{2015}$.

(A) 3 (B) 9 (C) 7 (D) 0 (E) 1

Solution: (D)

For any $k \geq 0$, the last digit of 3^{4k+1} is 3, the last digit of 3^{4k+2} is 9, the last digit of 3^{4k+3} is 7 and the last digit of 3^{4k} is 1. In the sum in question there are 504 terms of each of the form 3^{4k}, 3^{4k+1}, 3^{4k+2}, 3^{4k+3}, respectively. The last digit of the sum is the same as the last digit of $504 \cdot (3 + 9 + 1 + 7) = 10080$.

13. Evaluate $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\cdots\left(1-\dfrac{1}{2015^2}\right)$.

    (A) $\dfrac{1006}{2015}$ (B) $\dfrac{1007}{2015}$ (C) $\dfrac{1008}{2015}$ (D) $\dfrac{1009}{2015}$ (E) $\dfrac{202}{403}$

    Solution: (C)

    We have $1-\frac{1}{k^2}=\frac{k^2-1}{k^2}=\frac{(k-1)(k+1)}{k^2}$. As k ranges over $2,3,\ldots,2015$, the factors of the form $(k-1)$ contribute $2014!$, the factors of the form $(k+1)$ contribute $2016!/2$, and the factors k^2 contribute $(2015!)^2$. Canceling factors yields $\dfrac{2016}{2015\cdot 2}$.

14. A restaurant makes two strengths of iced tea. They start with one container of pure water and another of tea concentrate. The first step of mixing is to pour enough liquid from container A into container B to double the volume in container B. The contents of container B are thoroughly mixed and then enough liquid in container B is poured into container A in order to double the volume that was left in container A. After this mixing, the two containers have the same volume and the blend in container A consists of g gallons of pure water and $g+3$ gallons of tea concentrate. How many gallons of (the original) tea concentrate are in container A after the second mixing?

    (A) 7/2 (B) 9/2 (C) 11/2 (D) 13/2 (E) 15/2

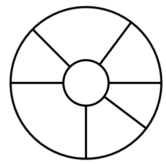
    Solution: (B)

    One way to solve is to first get a relation between the original volumes in A and B. Let x be the number of gallons in A and y the number of gallons in B. After the first mixing, A contains $x-y$ gallons and B contains $2y$ gallons. After the second mixing, A contains $2x-2y$ gallons and B contains $2y-(x-y)=3y-x$ gallons. Since the volumes are equal at this point, $3x=5y$. Thus A started with $5c$ gallons of liquid and B started with $3c$ gallons of liquid for some number c. After the second mixing, each contains $4c$ gallons of diluted tea.

    To find c, introduce another letter d with $d=c$ to help keep track of how the mixing occurs. Then repeat the mixing. Start with $5c$ in A and $3d$ in B. (I) The first mix leaves $2c$ in A and adds $3c$ to B, so now B contains $3c+3d$. (II) To add a total of $2c$ to A, add $c+d(=2c)$ from B into A, so now A contains $3c+d$ (total of $4c$) and B contains $2c+2d$ (total of $4c$). Since $c=d$ and A contains more tea concentrate than water, $5c$ is the original volume of tea concentrate and $3d=3c$ is the original volume of pure water. In addition, $3c=d+3=c+3$ so $c=3/2$. Thus after the second mixing, A contains 9/2 gallons of tea concentrate.

15. A round sign is divided into seven sections as in the diagram. The sign is to be painted with five colors (blue, green, orange, red and yellow) so that each section is painted with

one color and no adjacent sections are the same color. Also, each color must be used at least once and no color can be used more than twice. In how many different ways can the sign be painted?



(A) 1260 (B) 1480 (C) 2160 (D) 2520 (E) 5040

Solution: (C)

Two colors will be used twice each and the others will be used once each. There are 10 ways to make this choice. Note that the color in the center disk can be used only once. Choose one of the colors that is to be used twice (for example the one that comes first alphabetically for these two) and paint two sections with it. There are $6 \cdot 3/2 = 9$ ways to do this (for any one section, two (plus the center) are adjacent, thus there are $6 \cdot 3/2$ ways to choose two nonadjacent sections). For the other color to be used twice, there are 4 ways to color two nonadjacent sections from the four remaining. Finally for the three remaining colors & sections there are $3! = 6$ ways to finish painting the sign for a total of $10 \cdot 9 \cdot 4 \cdot 6 = 2160$.

16. Al wants to buy a new TV and the one he wants just went on sale for 25% off. Alas, Al forgot about the 8% sales tax and so when the TV is rung up, he finds he is \$35 short. The store manager comes by and says, "Give it to him for 30% off." With that Al can buy the TV and so he walks out with the TV and \$5.50 in his pocket. How much is the sales tax on Al's purchase?

(A) \$37.50 (B) \$40.50 (C) \$41.25 (D) \$42.00 (E) \$43.74

Solution: (D)

Let A be the amount of money Al has and let P be the original price of the TV. Then $0.75P + 0.08 \cdot 0.75P = A + 35$ and $0.7P + 0.08 \cdot 0.7P = A - 5.5$. Then $0.054P = 40.5$ so $P = 750$. Thus the 30% off price is \$525 and the sales tax is \$42.

17. For real numbers $a > b > 0$, let $P = (a, c)$ and $Q = (b, d)$ be points on the parabola given by $y = x^2$ and let $R = (a, f)$ and $S = (b, g)$ be points on the parabola given by $y = 4x^2$. If the segment $\overline{RS}$ is twice as long as $\overline{PQ}$, then what is the value of $a + b$?

(A) 1/4 (B) 1/2 (C) 1 (D) 2 (E) 4

Solution: (B)

$\sqrt{(a-b)^2+(4a^2-4b^2)^2}$ is the length of $\overline{RS}$ and $\sqrt{(a-b)^2+(a^2-b^2)^2}$ is the length of $\overline{PQ}$. Thus $(a-b)^2+16(a-b)^2(a+b)^2 = 4[(a-b)^2+(a-b)^2(a+b)^2]$, canceling the common factor of $(a-b)^2$ yields $1+16(a+b)^2 = 4+4(a+b)^2$. So $a+b=\sqrt{3/12}=1/2$.

18. The vertical line L given by $x=-2$ and the point $F=(4,3)$ determine a unique parabola in the plane. This parabola consists of the set of points $R=(f,g)$ where the distance from R to F is the same as the distance from R to the point $T=(-2,g)$ on the line L. In general, the line through R that is tangent to the parabola at R (it touches the parabola at R but does not cross over) is perpendicular to the line $\overleftrightarrow{FT}$. If the slope of $\overleftrightarrow{FT}$ is $1/2$, what is the x-coordinate of R?

(A) 1 (B) 3/2 (C) 7/4 (D) 9/4 (E) 5/2

Solution: (C)

The slope of $\overleftrightarrow{FT}$ is given by $\dfrac{3-g}{4-(-2)}=1/2$. So $3-g=3$. The distance from R to T is given by $f+2$ and the distance from R to F is given by $\sqrt{(f-4)^2+(g-3)^2} = \sqrt{(f-4)^2+9}$. Thus $(f+2)^2=(f-4)^2+9$, squaring and canceling f^2 from both sides yields $4f+4=-8f+16+9$, so $12f=25-4=21$ and $f=7/4$.

19. At the beginning of the year there were 25 students in class, an odd number of them was female. When 7 new students joined the class, the proportion of female students increased by 25%. How many female students are there in the class now?

(A) 6 (B) 7 (C) 8 (D) 12 (E) 16

Solution: (C)

Assume x female students were already in the class and y joined later. Since the proportion of female students increased by 25%, we get $\frac{x+y}{32}=\frac{5}{4}\cdot\frac{x}{25}$ or $y=3x/5$. Hence y is a multiple of 3 and it is an odd number. The only odd multiple of 3 that is less than or equal to 7 is 3. Therefore $y=3$, $x=5$, and the number of female students has become 8.

20. What is the value of the expression $\dfrac{1}{2!}+\dfrac{2}{3!}+\ldots+\dfrac{2014}{2015!}$? ($n!$ is the product of $1\cdot 2\cdot\ldots\cdot n$)

(A) $1-\frac{1}{2015!}$ (B) $1+\frac{1}{2015!}$ (C) $1-\frac{2014}{2015!}$ (D) $1+\frac{2014}{2015!}$ (E) $1-\frac{2014!}{2015!}$

Solution: (A)

Observe that $\dfrac{n}{(n+1)!}=\dfrac{n+1}{(n+1)!}-\dfrac{1}{(n+1)!}=\dfrac{1}{n!}-\dfrac{1}{(n+1)!}$. Then the given sum take the form $\left(\frac{1}{1!}-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{3!}-\frac{1}{4!}\right)+\ldots+\left(\frac{1}{2014!}-\frac{1}{2015!}\right)$. All terms in this sum are canceled except the first and the last ones.

21. The odometer on an automobile gives six-digit readings, one for each mileage from $000,000$ to $299,999$. A reading is "palindromic" if it reads the same from left to right and from right to left, for example $027,720$ and $163,361$. Find the total number of palindromic readings.

    (A) 100 (B) 200 (C) 300 (D) 400 (E) 500

    Solution: (C)

    We have to select a value for each of the three leftmost entries, then the remaining three entries are forced. There are three choices for the leftmost entry, ten for the next, and ten for the next, hence $3 \times 10 \times 10 = 300$ palindromic readings.

22. How many ways are there to pay one dollar, using nickels, dimes and quarters?

    (A) 26 (B) 29 (C) 30 (D) 31 (E) 32

    Solution: (B)

    Let x be the number of nickels, y the number of dimes, z the number of quarters used. We then need to find the number of nonnegative integer solutions of $5x+10y+25z = 100$. Dividing both sides by 5 we obtain the equivalent equation $x + 2y + 5z = 20$. Let us rearrange this as $x + 2y = 20 - 5z$. The possible values of z are $0, 1, 2, 3, 4$. The value of x can be any number that is at most $20 - 5z$ that has the same parity as $20 - 5z$, and the value of y is then uniquely determined. Thus, for $z = 0$, the possible values of x are $0, 2, \ldots, 20$ (11 solutions), for $z = 1$, the possible values of x are $1, 3, \ldots, 15$ (8 solutions), and so on, as summarized in the following table:

z	x	Number of solutions
0	$0, 2, \ldots, 20$	11
1	$1, 3, \ldots, 15$	8
2	$0, 2, \ldots, 10$	6
3	$1, 3, 5$	3
4	0	1

    The total number of all solutions is $11 + 8 + 6 + 3 + 1 = 29$

23. Which of the following numbers can be obtained by taking the difference of a 3-digit decimal number $n = abc_{10}$ with the number $n' = cba_{10}$, obtained by writing the digits of n in reverse order?

    (A) 9 (B) 97 (C) 297 (D) 400 (E) 496

    Solution: (C)

The difference of $n = 100a + 10b + c$ and $n' = 100c + 10b + a$ is $n - n' = 99(a - c)$, a multiple of 99. Among the numbers listed, only $297 = 3 \cdot 99$ is a multiple of 99. This one arises as a difference of 401 and 104, for example.

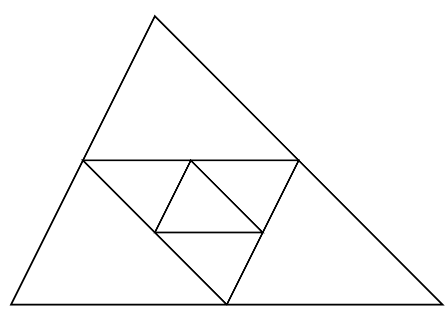
24. Cards are turned over one at a time from a well shuffled 52 card deck until the first heart appears. The probability that exactly 5 cards are required is in which of the following intervals?

    (A) $[0, .2]$ (B) $(.2, 4]$ (C) $(.4, .6]$ (D) $(.6, .8]$ (E) $(.8, 1]$

    Solution: (A)

    The probability is
    $$\frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \cdot \frac{36}{49} \cdot \frac{13}{48} \approx 0.082.$$

25. There are three triangles of different sizes: small, medium and large. The small one is inscribed in the medium one such that its vertices are at the midpoints of the three edges of the medium one. The medium triangle is inscribed in the large triangle in the same way as shown in the figure. If the small triangle has area 1, what is the sum of the areas of the three triangles?



    (A) 14 (B) 16 (C) 19 (D) 21 (E) 25

    Solution: (D)

    The larger triangle has area 4 times that of the inscribed one. So the medium triangle has area 4 and the large triangle has area 16 and the total area is $1 + 4 + 16 = 21$.