UNC Charlotte 2009 Comprehensive

March 9, 2009

1. Danica was driving in a 500 mile race. After 250 miles, Danica's average speed was 150 miles per hour. Approximately how fast should Danica drive the second half of the race if she wants to attain an overall average of 180 miles per hour?

   (A) 210 **(B)** 215 **(C)** 220 **(D)** 225 **(E)** 230

   Solution: D. Danica has driven for $250/150 = 5/3$ hours so far. If she drives the remaining 250 miles in x hours we need $\frac{500}{5/3+x} = 180$. This gives $x = 10/9$ so her average speed must be $250/x = 225$ mph.

2. Suppose a and b are positive numbers different from 1 satisfying $ab = a^b$ and $a/b = a^{2b}$. Then the value of $8a + 3b$ is

   (A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

   Solution: D. Squaring the first equation, we get $a^2b^2 = a^{2b}$ so that $a^2b^2 = a/b$ or $a = 1/b^3$. Substituting this into the first equation, we find $1/b^2 = 1/b^{3b}$. This gives $2 = 3b$ so that $b = 2/3$. Then $a = 27/8$.

3. Let a, b and c be positive numbers with both a and b greater than 1. Find the solution of the equation $\log_b x - \log_b(x - c) = a$.

   (A) $\dfrac{cb^a}{b^a - 1}$ **(B)** $\dfrac{ab^a}{1 - b^a}$ **(C)** $\dfrac{cb^a}{1 + b^a}$ **(D)** $\dfrac{ab^a}{1 + b^c}$ **(E)** $\dfrac{c}{b^a - 1}$

   Solution: A. Combining logarithms we have $\log_b\left(\frac{x}{x-c}\right) = a$ so that $\frac{x}{x-c} = b^a$ or $x = b^a(x - c)$. Therefore, $x = \frac{cb^a}{b^a-1}$.

4. Let x denote the smallest positive integer satisfying $12x = 25y^2$ for some positive integer y. What is $x + y$?

   (A) 75 **(B)** 79 **(C)** 81 **(D)** 83 **(E)** 88

   Solution: C. Note that $25y^2 = (5y)^2$ so $12x = 2^2 \cdot 3x$ must be a perfect square multiple of 5. The smallest integer $3x$ is $3^2 \cdot 5^2$, so $x = 3 \cdot 5^2 = 75$ and in this case $y = 6$. Thus $x + y = 81$.

5. What is the area of the triangular region in the first quadrant bounded on the left by the y-axis, bounded above by the line $7x + 4y = 168$ and bounded below by the line $5x + 3y = 121$?

   (A) 16 **(B)** 50/3 **(C)** 17 **(D)** 52/3 **(E)** 53/3

   Solution: B. The triangle has a base along the y-axis of $42 - 121/3 = 5/3$ and an altitude of 20 (the lines intersect at $(20, 7)$. So the area is $\frac{1}{2} \cdot \frac{5}{3} \cdot 20 = 50/3$.

6. We want to divide the L shaped region shown in Figure **??** into two pieces with equal areas by means of a line from P to Q. The point P is always in the upper left hand corner of the region and the point Q must lie along the bottom edge as shown. When this is done, which of the following numbers is closest to the distance from A to Q?

   (A) 1.2 **(B)** 1.3 **(C)** 1.4 **(D)** 1.5 **(E)** 1.6

Figure 1: Illustration to Question **??**.

Solution: D. Place Q 1.5 units to the right of A. Then remove the rectangle $BCDQ$ and place it in the position $FGBH$ as shown in Figure **??**. The line PQ cuts the area in half.

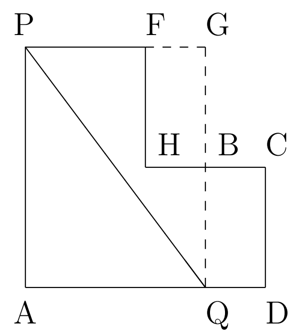

Figure 2: Illustration to the solution of Question **??**.

7. Let $N = \underline{abcdef}$ be a six-digit number such that $\underline{defabc}$ is six times the value of $\underline{abcdef}$. What is the sum of the digits of N?

   (A) 27 **(B)** 29 **(C)** 31 **(D)** 33 **(E)** 35

Solution: A. Let $x = abc$ and $y = def$. Then $1000y + x = 6(1000x + y)$, from which it follows that $857x = 142y$. Since 857 and 142 are relatively prime, 857 must divide y. Since y is a three-digit number, y must be 857, and $x = 142$. The sum of the digits of N is $8 + 5 + 7 + 1 + 4 + 2 = 27$.

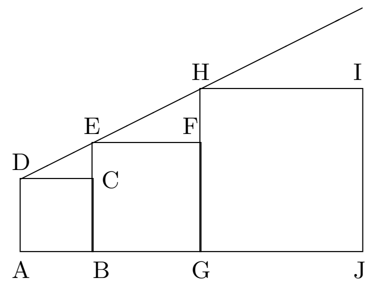

Figure 3: Illustration to Question **??**.

8. Three adjacent squares rest on a line. Line L passes through a corner of each square as shown in Figure **??**. The lengths of the sides of the two smaller squares are 4 cm and 6 cm. Find the length of one side of the largest square.

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 14

Solution: B. The slope of the line in question is $1/2$, so the largest square is 9×9.

9. Given that (x, y) satisfies $x^2 + y^2 = 9$, what is the largest possible value of $x^2 + 3y^2 + 4x$?

   (A) 22 **(B)** 24 **(C)** 36 **(D)** 27 **(E)** 29

   Solution: E. Replace y^2 with $9-x^2$ to get $x^2-3x^2+27+4x = 29-2x^2+4x-2 = 29 - 2(x-1)^2$, which is at most 29.

10. Two red, two white, and two blue faces, all unit squares, are available for building a cube. How many distinguishable cubes can be built?

    (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

    Solution: B. First, suppose that there is exactly one pair of opposite faces of the same color. In this case, there is just one cube with the blue faces opposite, one with the white faces opposite, and one with the red faces opposite. There are no cubes with exactly two pairs of opposite faces of the same color. There is just one cube where all three pairs of opposite faces are the same color. Finally, there are two cubes where no pair of opposite faces are the same color. We get a total of six distinguishable cubes. Thanks to J. Parker Garrison for correcting this solution. We failed to take into account that the last case includes both a 'left handed' and a 'right handed' cube.

11. Seven women and seven men attend a party. At this party, each man shakes hands with each other person once. Each woman shakes hands only with men. How many handshakes took place at the party?

    (A) 49 **(B)** 70 **(C)** 91 **(D)** 133 **(E)** 182

    Solution: B. Seven men can shake hands with each other in $7 \cdot 6/2 = 21$ ways. The men can shake hands with the women in $7 \cdot 7 = 49$ ways. Adding these, we get 70 handshakes.

12. How many different sums can you get by adding three different numbers from the set $\{3, 6, 9, \ldots, 21, 24\}$?

    (A) 15 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 22

    Solution: B. The smallest possible sum is $3 + 6 + 9 = 18$ and the largest is $18 + 21 + 24 = 63$ and the set of all possible sums is $\{18, 21, 24, \ldots, 63\}$, that is, the set of multiples of 3 in the range 18 up to 63. There are 16 numbers in this set.

13. Two different unit squares are randomly selected from the 16 squares in the 4×4 grid shown in Figure **??**. What is the probability that they have at least one point in common?

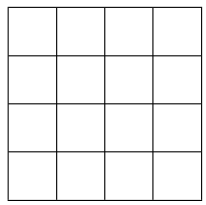

Figure 4: Illustration to Question **??**.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{20}$ **(D)** $\frac{11}{30}$ **(E)** $\frac{3}{8}$

Solution: C. Each corner square touches 3 others. Each other edge square touches 5 others, and each interior square touches 8 others. Therefore there are 42 pairs of squares that touch one-another. There are $\binom{16}{2} = 120$ pairs of squares, so the probability is $\frac{42}{120} = \frac{7}{20}$.

14. A doodlebug is an insect which crawls among the lattice points (points with integer coordinates) of the plane. Each move of a doodlebug is 5 units horizontally or vertically followed by 3 units in a perpendicular direction. For example, from (0,0) a doodlebug can move to any of the eight locations $(\pm 5, \pm 3), (\pm 3, \pm 5)$. What is the fewest number of moves required to get from $(0, 0)$ to $(8, 0)$?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** No such sequence of moves exists

    Solution: C. Note that 8 is not a multiple of 5 or a multiple of 3, so the horizontal portion of the moves must contain at least one three and one five. This must then be true of the vertical portion of the moves as well. Since the y-coordinate of (8,0) is zero, we need a multiple of fives and a multiple of threes to add to zero. The least number of moves that allow this requires 5 three's and 3 fives. This works since we can choose the vertical portion to be $5 + 5 + 5 - 3 - 3 - 3 - 3 - 3 = 0$ and the horizontal portion to be $3 + 3 - 3 + 5 + 5 - 5 + 5 - 5 = 8$.

15. Charlie's current age is a prime number less than 100. The product of the digits of Charlie's age is the same number as it was seven years ago. In how many years will the product of the digits be the same again?

    (A) 7 **(B)** 8 **(C)** 9 **(D)** 11 **(E)** 13

    Solution: C. Charlie's current age must be a two digit number. Suppose that Charlie's current age is $10a + b$ where a, b are one of the digits $0, \dots, 9$ $(a \neq 0)$. Then seven years ago, his age would be $10a + b - 7$. If $b - 7 \geq 0$ then the product of the digits can not be ab as it was. Therefore we can write $10a+b-7 = 10(a-1)+b+3$ and the product of the digits $(a-1)(b+3)$ must equal ab. This gives $b = 3a - 3$. Since $0 \leq b < 7$ we can only have $a = 2$ or 3 and current ages of 23 and 36. Only 23 is prime. The next time the product of the digits will be six is at age 32, which is 9 years later.

16. Which of the following numbers is closest to the value of the continued fraction $\dfrac{2}{1+\frac{2}{1+\frac{2}{1+\ldots}}}$

    (A) 1.00 **(B)** 1.02 **(C)** 1.04 **(D)** 1.06 **(E)** 1.08

    Solution: A. Let x be the value of the continued fraction. Then $x = \frac{2}{1+x}$ so that $x = 1$

17. Suppose that $f(n) = 2f(n+1) - f(n-1)$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ and $f(1) = 4$ and $f(-1) = 2$. Evaluate $f(2)$.

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: E. We have $f(0) = 2f(1) - f(-1) = 8 - 2 = 6$. Substituting $f(0) = 6$ into $f(1) = 2f(2) - f(0)$ gives $4 = 2f(2) - 6$. Thus $f(2) = 5$.

18. Each letter in the long division below stands for a single digit of a decimal number. The letter M is not zero and no leading digit is zero. Different letters may be used for the same digit. What is the value of B?

					J	K	8	L	M
A	B	)	C	D	E	F	G	H	I
			N	P	Q				
						R	S		
						T	U		
							V	W	X
							Y	O	Z
									1

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: B. Because the quotient has 5 digits but only 3 subtractions were performed, 2 digits of the quotient must be zero. These digits can not be the first or last. So $K = L = 0$. We see J times AB is NPQ, but 8 times AB is TU, so J must be 9. M times AB is YOZ, a three digit number, so M is also 9. We know AB times 8 has a two digit product, while AB times 9 has a three digit product. Therefore $AB = 12$. Since $CDEFGHI$ equals AB times $JK8LM$ plus one, we find the dividend to be 1089709.

19. Which of the following numbers is the sum of the squares of three consecutive odd numbers?

    (A) 1281 **(B)** 1441 **(C)** 1595 **(D)** 1693 **(E)** 1757

    Solution: C. Let u be the middle number. Then the sum is $(u-2)^2 + u^2 + (u+2)^2 = 3u^2 + 8$. If the sum is S then $u = \sqrt{\frac{S-8}{3}}$ is an integer. This is only the case for option (C).

20. If $(2^x - 4^x)^2 + (2^x + 4^x)^2 = 144$, what is x?

    (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{4}$ **(E)** $\frac{3}{2}$

    Solution: E. To simplify the calculation, let $t = 2^x$. Then

    $$144 = (t - t^2)^2 + (t + t^2)^2 = 2(t^2 + t^4)$$

    Thus $t^4 + t^2 - 72 = (t^2 + 9)(t^2 - 8) = 0$ and hence $t^2 = 2^{2x} = 8 = 2^3$. Therefore $x = 3/2$.

21. An urn contains marbles of four colors, red, green, blue and yellow. All but 25 are red, all but 25 are yellow, and all but 25 are blue. All but 36 are green. How many of the marbles are green?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: A. Let r, g, y, b denote the number of red, green, yellow and blue marbles respectively. Then the equations translate into $g+b+y = 25$, $r+g+y = 25$, $r+b+g = 25$ and $r+b+y = 36$. Adding all four equations together yields $3(r+g+y+b) = 111$, which means that $r+g+y+b = 37$. Subtract the fourth equation from this one to get $g = 1$.

22. How many two-element subsets $\{a, b\}$ of $\{1, 2, 3, \ldots, 16\}$ satisfy ab is a perfect square?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    Solution: E. Any two-element subset of $\{1, 4, 9, 16\}$ satisfies the condition. There are six of these. There are just two other sets, $\{2, 8\}$ and $\{3, 12\}$.