Symmetry

1. Find the volume of a rectangular box whose left side, front side, and bottom have areas of 10 square inches 15 square inches and 294 square inches, respectively.

Solution: Let $w$, $h$, and $l$ denote the three dimensions. Then the given can be written $wh = 10$, $wl = 15$, and $lw = 294$. Taking products of all three gives $w^2h^2l^2 = 10 \cdot 15 \cdot 294$. It follows that Volume $= whl = \sqrt{10 \cdot 15 \cdot 294} = 210$.

2. Arrange the numbers 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 in the ten locations so that the sum of the four numbers along each of the five lines is the same.

Solution: There are many solutions, but only one set of four element subsets that can serve as lines. Note that we can find five equations, each of the form $x_i + x_j + x_k + x_l = K$, where $K$ is the constant line sum. Adding all five equations enables us to solve for $K$. We get $K = 24$. We then search for the five subsets $L_1$, $L_2$, $L_3$, $L_4$, and $L_5$ of $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}$ satisfying $L_i \cap L_j$ consists of a single number. The sets $\{1, 3, 8, 12\}$, $\{1, 4, 9, 10\}$, $\{2, 4, 6, 12\}$, $\{2, 5, 8, 9\}$, and $\{3, 5, 6, 10\}$ are the only sets that can work.
Symmetry

4. Solve simultaneously:

\[
\begin{align*}
  x + 2y + z &= 14 \\
  2x + y + z &= 12 \\
  x + y + 2z &= 18
\end{align*}
\]

**Solution:** Add the three equations together to get \(4x + 4y + 4z = 44\), so \(x + y + z = 11\). Subtract this from each of the given equations, one at a time to get \(x = 1\), \(y = 3\), and \(z = 7\).

5. Solve simultaneously:

\[
\begin{align*}
  x + y &= 7 \\
  x + z &= -2 \\
  y + z &= 9
\end{align*}
\]

**Solution:** Add the three together to get \(2(x + y + z) = 14\), and solve just like the one above.

6. Solve simultaneously:

\[
\begin{align*}
  xy &= -6 \\
  yz &= -2 \\
  xz &= 12
\end{align*}
\]

**Solution:** Multiply the three equations to get \(x^2y^2z^2 = 144\) from which it follows that \(xyz = \pm 12\). These two possibilities lead to two sets of answers.

7. Solve simultaneously:

\[
\begin{align*}
  (x + 1)(y + 1) &= 24 \\
  (y + 1)(z + 1) &= 30 \\
  (x + 1)(z + 1) &= 20
\end{align*}
\]

**Solution:** Again multiply the three equations to get \((x+1)^2(y+1)^2(z+1)^2 = 24 \cdot 30 \cdot 20\) from which it follows that \((x+1)(y+1)(z+1) = \sqrt{24 \cdot 30 \cdot 20} = \pm 120\). These two possibilities lead to two sets of answers.
Symmetry

8. Solve simultaneously:

\[ xy - x - y = 11 \]
\[ yz - y - z = 14 \]
\[ xz - x - z = 19 \]

**Solution:** Add one to both sides of each equation and factor to get

\[ xy - x - y + 1 = (x - 1)(y - 1) = 12 \]
\[ yz - y - z + 1 = (y - 1)(z - 1) = 15 \]
\[ xz - x - z + 1 = (x - 1)(z - 1) = 20 \]

Then solve the system just as we did the one above.

9. Solve simultaneously:

\[ x(x + y + z) = 4 \]
\[ y(x + y + z) = 6 \]
\[ z(x + y + z) = 54 \]

**Solution:** Add the equations to get \((x + y + z)^2 = 64\), so \(x + y + z = \pm 8\). These lead to two solutions, \((x, y, z) = (0.5, 0.75, 6.75)\) and \((x, y, z) = (-0.5, -0.75, -6.75)\).

10. Solve simultaneously:

\[ x + \lfloor y \rfloor + \langle z \rangle = 1.1 \]
\[ \langle x \rangle + y + \lfloor z \rfloor = 2.2 \]
\[ \lfloor x \rfloor + \langle y \rangle + z = 3.3 \]

The notation \([x]\) is read ‘floor of \(x\)’ and means the largest integer not greater than \(x\). The notation \(\langle x \rangle\) is read ‘fractional part of \(x\)’ and means the \(x - \lfloor x \rfloor\).

**Solution:** Add the three equations to get \(2(x + y + z) = 6.6\). Then subtract the first equation from \(x + y + z = 3.3\) to get \(y - \lfloor y \rfloor + z - \langle z \rangle = \langle y \rangle + \lfloor z \rfloor = 2.2\). But \(\lfloor z \rfloor\) must be the integer part of \(\langle y \rangle + \lfloor z \rfloor\), so \(\langle y \rangle = 0.2\) and \(\lfloor z \rfloor = 2\). The other values \(x = 1.0\), \(y = 0.2\) and \(z = 2.1\) follow from this.
Symmetry

11. Given that $a$ is a real number, solve simultaneously:

\begin{align*}
  x^2 - xy &= a \\
  y^2 - xy &= a(a - 1).
\end{align*}

**Solution:** There are two possibilities, $x = 1, y = 1-a$ and $x = -1, y = -1-a$.

12. Solve simultaneously:

\begin{align*}
  x_2 + x_3 + x_4 + \ldots + x_n &= 1 \\
  x_1 + x_3 + x_4 + \ldots + x_n &= 2 \\
  x_1 + x_2 + x_4 + \ldots + x_n &= 3 \\
  \vdots \\
  x_1 + x_2 + x_3 + \ldots + x_{n-1} &= n.
\end{align*}

**Solution:** Add all the equations together and then subtract.

13. A triangle has sides of lengths 13, 14, and 15. Its inscribed circle divides each side into two segments, making six altogether. Find the length of each segment.

**Solution:** Let $x, y,$ and $z$ denote the lengths of the segments. Then, in some order $x + y = 13, y + z = 14,$ and $x + z = 15,$ which gives $x = 7, y = 6, and z = 8.$

14. Solve simultaneously:

\begin{align*}
  xy + xz &= 13 \\
  xz + yz &= 25 \\
  xy + yz &= 20.
\end{align*}

**Solution:** As before, add the three equations and then subtract one at a time to get $yz = 16, xy = 4, and xz = 9.$ Then solve that system by multiplying to get $x^2y^2z^2 = 24^2,$ or $xyz = \pm 24$ from which we get the two solutions $(x, y, z) = (3/2, 8/3, 6)$ and $(-3/2, -8/3, -6).$
Symmetry

15. Solve simultaneously:

\[ 2x_1 + x_2 + x_3 + x_4 + x_5 = 6 \]

\[ x_1 + 2x_2 + x_3 + x_4 + x_5 = 12 \]

\[ x_1 + x_2 + 2x_3 + x_4 + x_5 = 24 \]

\[ x_1 + x_2 + x_3 + 2x_4 + x_5 = 48 \]

\[ x_1 + x_2 + x_3 + x_4 + 2x_5 = 96 \]

**Solution:** Again add all the equations together and subtract to get \( x_1 = -25, x_2 = -19, x_3 = -7, x_4 = 17 \) and \( x_5 = 65 \).

16. Solve the system of equations:

\[ \frac{xyz}{x + y} = 7.2, \quad \frac{xyz}{y + z} = 4, \quad \frac{xyz}{x + z} = 4.5. \]

**Solution:** Take reciprocals and find that \( x + y + z = 11/36 \). This eventually leads to \( x = 2, y = 3, \) and \( z = 6 \).

17. Solve the equation

\[ \frac{x - 3}{2001} + \frac{x - 5}{1999} + \frac{x - 7}{1997} + \frac{x - 9}{1995} = \frac{x - 2000}{4} + \frac{x - 1998}{6} + \frac{x - 1996}{8} + \frac{x - 1994}{10}. \]

**Solution:** See if you can make each side have a value of 4 by making each term have value 1. Then note that the equation is linear, so it can have at most one solution.