This is the seventeenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I’m looking for problems with solutions that don’t depend on highly technical ideas. Ideal problems should be easily understood and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at hbreiter@email.uncc.edu. In general, we’ll list the problems in one issue and their solutions in the next issue.

17.1 Let \( f(x) = x^2 + 8x + 12 \). Find all real solutions of the equation \( f(f(f(f(x)))) = 0 \).

17.2 A bug starts from the origin in the plane and crawls to \((1, 1)\) after one minute. After another minute it crawls to \((0, 1)\). Consider the counterclockwise spiral path, shown below, starting at the origin. Each unit segment between lattice points take exactly one minute to traverse and each diagonal segment of length \( \sqrt{2} \) also takes one minute.

(a) Where is the bug after exactly 2008 minutes?

(b) How many minutes does it take for the bug to get to the ordered pair \((19, 99)\)?
16.1 (University of South Carolina Math Contest, 2006) Find a real function $f$ with domain all of $\mathbb{R}$ except possibly two points such that

$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$ 

**Solution:** Note that the given condition implies

$$a.f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x},$$

and

$$b.f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x}.$$

Taking the sum of the given equation with $b$ and subtracting $a$ yields $2f(x) = x - \frac{1}{1-x} + \frac{x-1}{x}$, which is equivalent to

$$f(x) = \frac{-x^3 + x - 1}{2x(1-x)}.$$ 

16.2 The number $N = 37! = 1 \cdot 2 \cdot 3 \cdots 37$ is a 44-digit number. The first 33 digits are $K = 137637530912263450463159795815809$. In fact, $N = K \cdot 10^{11} + L \cdot 10^8$, where $L$ is a 3-digit number. Find the three-digit number $L$.

**Solution:** $37! = 13,763,753,091,226,345,046,315,979,581,580,902,400,000,000$. So the number $L$ is 24. Let $L = abc$. There are three solutions. One depends on the fact that $1001 = 7 \cdot 11 \cdot 13$. Write 37! in base 1000, so that each digit is of the form $uvw$, where $u, v,$ and $w$ are decimal digits. Thus $37! = 13 \cdot (1000)^{14} + 763 \cdot (1000)^{13} + \cdots + 9ab \cdot (1000)^3 + c00 \cdot 1000^2$. But since $1000^n = (1001 - 1)^n = 1001^n - n \cdot 1001^{n-1} + \cdots + (-1)^n$, by the binomial theorem, we can see that only the last term fails to be a multiple of 1001. This means that $uvw \cdot 1000^n \equiv (-1)^n$ (mod 1001). Thus $37! \equiv 13 + 753 + 226 + 46 + 979 + 580 + 100c - (763 + 91 + 345 + 315 + 581 + 900 + 10a + b) \equiv -398 + 100c - 10a - b \equiv 0$ (mod 1001). This can happen only when $c = 4, a = 0$ and $b = 2$, so $L = 24$.

The second method depends upon the fact that $999 = 27 \cdot 37$, so 37! is a multiple of 999. Once again, express the number 37! in base 1000, where $L = abc$. Since 1000 $\equiv 1$ (mod 999), the number 37! is the sum of its base 1000 digits, mod 999. Hence, $37! \equiv 13 + 763 + 753 + 091 + 226 + 345 + 046 + 315 + 979 + 581 + 580 + 9ab + c00 + 000 + 000 \equiv 5592 + 9ab + c00 \equiv 0$ (mod 999). The only way this can happen is $5592 + 9ab + c00 = 6 \cdot 999 = 5994$, from which it follows that $ab + c00 = 402$, which means $c = 4, a = 0$ and $b = 2$.

The third and easiest method to understand uses brute force to get the rightmost digit, and then mod 9 and mod 11 arithmetic to find the other two digits.