UNC Charlotte 2005 Algebra

March 7, 2005

1. What is the y-coordinate of the vertex of the parabola $y = x^2 - 6x + 4$?

   (A) -5 **(B)** -3 **(C)** 3 **(D)** 5 **(E)** 9

   Solution: A. Complete the square to get $y+9 = x^2-6x+9+4$, so $y+5 = (x-3)^2$, and $y = (x-3)^2 - 5$, so the vertex is $(3, -5)$.

2. The line $y = x$ intersects the circle $(x-3)^2 + (y-2)^2 = 1$ in two points. What is the sum of the x-coordinates of the two points?

   (A) 3 **(B)** 4 **(C)** 9/2 **(D)** 5 **(E)** 11/2

   Solution: D. Replacing y with x yields $(x-3)^2 + (x-2)^2 = 1$ which reduces to $2x^2 - 10x + 12 = 0$, the sum of whose zeros is $-(-10/2) = 5$. Alternatively, solve the quadratic for $x = 2$ and $x = 3$.

   For yet another solution, notice that the center of the circle is the point $(3, 2)$ and the radius is 1. Since $3-3 = 0$ and $3-2 = 1$, one of the points is $(3, 3)$. The other is $(2, 2)$ since $(2-3)^2 = 1$ and $2-2 = 0$. Thus the sum is 5.

3. The mean of the 7-member list $x, 3x-4, 4x-3, -16, 9, 5x+2$, and $x-2$ is 4. What is the median of the list?

   (A) 5 **(B)** 6 **(C)** 6.5 **(D)** 8 **(E)** 9

   Solution: A. The sum of the seven is $7x - 7 = 7(x-1)$ which is 28 because the mean is 4. It follows that $x = 3$ and the list in order by value is $-16, 1, 3, 5, 9, 9, 17$, so the median is 5.

4. Suppose we have two thermometers, one which measures temperature F in degrees Fahrenheit and one which measures temperature C in degrees Celsius. The readings are related by the formula $F = \frac{9}{5}C + 32$. Is it possible to be at a location where both thermometers give the same numerical reading? If so, which of the locations below is the most likely place for this to happen?

   (A) There is no such location

   (B) at the North Pole (very cold)

   (C) at the equator (very hot)

   (D) in North Carolina (moderate temperature)

   (E) all of these locations are equally likely

   Solution: B. $C = 9C/5 + 32$ implies $4C/5 = -32$ which implies that $C = -32 \cdot 5/4 = -40$.

   Alternatively, here is a simple way to eliminate choices C, D and E: if $C \geq 0$, then $F = (9/5)C + 32 > (9/5)C$. So $C < 0$ (out of my "moderate" range). To get to B as the answer: estimate where the lines $F = (9/5)C + 32$ and $F = C$ cross. For $C = 0$, $F = 32$ on the first line and $F = 0$ on the second. So the line $F = (9/5)C + 32$ is "above" the line $F = C$ when $C = 0$. Try multiples of -10 for C until you reach $C = -40$ where the exact crossing occurs.

5. During one quarter of the school year, Kristen took 3 tests. On the second test her grade increased by 25% compared to the first test and then on the third test her grade decreased by 23% compared to the second test. Then with respect to the first test, her grade on the third test

   (A) increased by about 2% **(B)** increased by about 1%

   (C) decreased by about 2% **(D)** decreased by about 4%

   (E) stayed almost the same

   Solution: D. Let x be the grade on Kristen's first test, then $1.25x$ is the grade on her second test, and $.77(1.25x) = .9625x$ is the third test grade. So the grade has decreased by almost 4%.

6. Find the largest possible integer n such that $1 + 2 + 3 + \cdots n \leq 200$.

   (A) 14 **(B)** 17 **(C)** 19 **(D)** 21 **(E)** 23

   Solution: C. The sum $1 + 2 + 3 + \cdots + n$ can be found easily to be $n(n + 1)/2$. Thus $n(n + 1)/2 \leq 200$ and $n(n + 1) \leq 400$. So n is close to 20. Check $n = 20$ to see that $1 + 2 + 3 + \cdot + 20 = 10 \cdot 21 = 210$, so we try $n = 19$ and see that $1 + 2 + \cdots + 19 = 190$.

   On the other hand, since $1 + 2 + \cdots + n = n(n + 1)/2 \leq 200$ it follows that $n(n + 1) = n^2 + n \leq 400$. No matter what the value of n, $n^2 < n^2 + n < (n + 1)^2$. Since $400 = 20^2$, n cannot be 20, but $n + 1$ can. So the correct value is 19.

7. The polynomial $p(x) = 2x^4 - x^3 - 7x^2 + ax + b$ is divisible by $x^2 - 2x - 3$ for certain values of a and b. What is the sum of a and b?

   (A) -34 **(B)** -30 **(C)** -26 **(D)** -18 **(E)** 30

   Solution: A. Because $x^2 - 2x - 3 = (x - 3)(x + 1)$ $p(x)$ has zeros of 3 and -1, $p(3) = 72 + 3a + b = 0$ and $p(-1) = -4 - a + b = 0$ which we can solve simultaneously to get $a = -19$ and $b = -15$.

   Alternatively, simply start by dividing $2x^4 - x^3 - 7x^2 + ax + b$ by $x^2 - 2x - 3$ by long division. The "last step" is to subtract $5x^2 - 10x - 15$ from $5x^2 + (9 + a)x + b$. The difference must be 0. So $b = -15$ and $9 + a = -10$. Thus $a = -19$ and $a + b + -34$. No factoring is done.

   Yet another method is to factor $x^2 - 2x - 3$ and see that it has two zeros, $x = 3$ and $x = -1$. Use synthetic division (starting with either zero–here starting with $x = -1$) to see that (1) $b - a - 4 = 0$, and (2) $2x^4 - x^3 - 7x^2 + ax + b = (x + 1)(2x^3 - 3x^2 - 4x + a + 4)$. Now use synthetic division with $x = 3$ on the quotient $2x^3 - 3x^2 - 4x + a + 4$ to see that $a + 19 = 0$. Thus $a = -19$. Put this into (1) to see that $b = -15$, etc.

8. The numbers x and y satisfy $2^x = 15$ and $15^y = 32$. What is the value of xy?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** none of **A**, **B**, **C** or **D**

   Solution: C. Note that $(2^x)^y = 15^y = 32$ so $2^{xy} = 2^5$ and $xy = 5$. Alternatively, $x = \log 15/\log 2$ and $y = \log 32/\log 15$ so $xy = \log 32/\log 2 = 5$.

9. One hundred monkeys have 100 apples to divide. Each adult gets three apples while three children share one. How many adult monkeys are there?

   (A) 10 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 33

   Solution: C. Assume there are x adults and y children, then $x + y = 100$ and $3x + (1/3)y = 100$. Solving simultaneously leads to $x = 25$ and $y = 75$.

10. Let x and y be positive integers satisfying

$$\frac{1}{x+1} + \frac{1}{y-1} = 5/6.$$

   Find $x + y$.

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

   Solution: D. Since x and y are natural numbers, $y \geq 2$. Let $u = x + 1$ and $v = y - 1$, u and v are natural numbers with $u \geq 2$ and $v \geq 1$. One of $1/u$ and $1/v$ is at least half of $5/6$, so either $u \leq 12/5 = 2.4$ or $v \leq 12/5 = 2.4$. Consider the two cases: case one, $u = 2$ so $x = 1$ then $v = 3$ then $y = 4$; case two, $v = 2$ so $y = 3$ then $u = 3$ then $x = 2$. In either case, $x + y = 5$.

11. Let x and y be two integers that satisfy all of the following properties:

   (a) $5 < x < y$,

   (b) x is a power of a prime and y is a power of a prime, and

   (c) the quantities $xy + 3$ and $xy - 3$ are both primes.

Among all the solutions, let (x, y) be the one with the smallest product. Which of the following statements is true?

(A) $x + y$ is a perfect square **(B)** the number xy is prime

(C) $y = x + 3$ **(D)** $y = x + 1$ **(E)** $x + y = 17$

Solution: D. From the condition (a), we see that $x \geq 6$ and $y \geq 7$, so that $xy - 3 \geq 39$. The first few prime numbers are larger that 39 are $43, 47, 53, 59, 61, \ldots$. Because of condition (c), we look for a pair that differs by 6. See the table on the last page. We can see that the first such pair is 47 and 53, which would give $xy = 50 = 2 \cdot 5^2$. We can't satisfy both (a) and (b) with this product. The next pair to check is 53 and 59. This gives $xy = 56 = 7 \cdot 2^3$ and the pair $(x, y) = (7, 2^3)$ satisfies (a)-(c).

Alternatively, (i) start a list of the integers larger than 5 that are powers of primes, 7, 8, 9, 11, 13, 16, 17,...; (ii) note that the smallest product with $x < y$ is $7 \cdot 8$ which just happens to be exactly 3 more than the prime 53 and 3 less than the prime 59.

12. The product of three consecutive positive integers is eight times their sum. What is the sum of their squares?

(A) 50 **(B)** 77 **(C)** 110 **(D)** 149 **(E)** 194

Solution: B. If the numbers are $n - 1, n$, and $n + 1$, then $(n - 1)(n)(n + 1) = 8(n - 1 + n + n + 1) = 24n$. Because $n \neq 0$, it follows that so $n^2 - 1 = 24$ and it follows that $n = \pm 5$. The sum of the squares is $16 + 25 + 36 = 77$.

13. If $ab = 999$ and both a and b are two digit positive integers, what is $a + b$?

(A) 64 **(B)** 66 **(C)** 120 **(D)** 336 **(E)** 1000

Solution: A. Clearly $999 = 9 \cdot 111$, but 9 has one to few digits and 111 has one too many. Since $1 + 1 + 1 = 3$, 3 is a factor of 111. Dividing yields $111 = 3 \cdot 37$, with 37 a prime. So the only factoring into two digit integers is $27 \cdot 37$.

14. Let $P(x)$ be a polynomial with $P(1) = 1$ and $P(x) = P(x-1) + x^3$ for all real x. Calculate $P(-3)$.

(A) -36 **(B)** -9 **(C)** 1 **(D)** 9 **(E)** 36

Solution: D. Note that $P(x-1) = P(x) - x^3$ for all x. Hence $P(0) = P(1) - 1 = 0$ and $P(-1) = P(0) - 0^3 = 0$. Now one can proceed to $P(-3) = P(-2) - (-2)^3 = P(-1) - (-1)^3 - (-2)^3 = 1^3 + 2^3 = 9$.

15. Solve the equation $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3$ for x.

(A) 4/3 **(B)** 5/3 **(C)** 7/5 **(D)** 9/5 **(E)** none of **A**, **B**, **C** or **D**

Solution: B. Cross multiplying and simplifying, we obtain $2\sqrt{x-1} = \sqrt{x+1}$. Squaring both sides yields $4(x-1) = x+1$ or $3x = 5$. Thus, $x = 5/3$.

16. Which of the following is closest to the smallest positive rational number that is an integer multiple of the numbers $\frac{10}{21}$, $\frac{5}{14}$, and $\frac{6}{7}$?

(A) 1.43 **(B)** 2.79 **(C)** 3.43 **(D)** 4.29 **(E)** 5.74

Solution: D. We can write the three numbers as 20/42, 15/42, 36/42. Since $20 = 2^2 \cdot 5$, $15 = 3 \cdot 5$. and, $36 = 2^2 \cdot 3^2$. The least common multiple is $(2^2 3^2 5)/42 = 30/7 \approx 4.29$.

17. Suppose all three of the points $(-2, 10), (1, -8)$, and $(4, 10)$ lie on the graph of $y = ax^2 + bx + c$. What is abc?

(A) -24 **(B)** 0 **(C)** 12 **(D)** 24 **(E)** 48

Solution: E. By symmetry, the vertex must be the point $(x, y) = ((-2+4)/2, y) = (1, y) = (1, -8)$, so $-b/2a = 1$. Evaluating at 1 yields $a \cdot 1^2 - 2a \cdot 1 + c = -8$. Replacing b and c with their values in terms of a, $a(-2)^2 - 2a(-2) + a - 8 = 10$ from which it follows that $a = 2$, whence $b = -4$ and $c = -6$.Thus the product is 48.

Alternatively, write the three conditions

$$4a - 2b + c = 10 \quad (1)$$
$$a + b + c = -8 \quad (2)$$
$$16a + 4b + c = 10 \quad (3)$$

Subtracting (2) from (1), we have $3a - 3b = 18$ so $a - b = 6$. Subtracting (1) from (3) gives $12a + 6b = 0$ so $2a + b = 0$. We now find $3a = 6$ so $a = 2$, then $b = a - 6 = -4$, and $c = -8 - a - b = -6$ and their product is 48.

Alternate solution using properties of parabolas. (1) First step is the same, use symmetry to get that the x-coordinate of the vertex is $x = 1$. (2) Next note that the point $(4, 10)$ is 3 units to the right of the vertex and $18 = 2 \cdot 3^2$ units above the vertex, this implies that $a = 2$. (3) Since $-b/2a = 1$, $b = -4$. Plugging in any of the three x-coordinates and calculating to get the corresponding y-coordinate yields $c = -6$.

18. An amount of \$2000 is invested at $r\%$ interest compounded continuously. After four years, the account has grown to \$2800. Assuming that it continues to grow at this rate for 16 more years, how much will be in the account?

(A) \$8976.47 **(B)** \$9874.23 **(C)** \$10001.99

(D) \$10756.48 **(E)** \$2004.35

Solution: D. Use the formula $A = Pe^{rt}$ where r is the annual rate of interest, t is the time in years, P is the principle, and A is the amount in the account at time t. Then $2800 = 2000e^{4r}$, which implies that $r = (\log 1.4)/4$. So the amount in the account after $4+16 = 20$ years is $A = 2000e^{20r} = 2000 \cdot (e^{\ln 1.4/4})^{20} = 2000(1.4)^5 = 10756.48$.

Alternate solution: Since 16 is an integer multiple of 4, all that matters is that we start with \$2000 and have \$2800 after four years–the type of compounding doesn't matter. Based on compounding on a four year cycle, the (decimal) rate is $800/2000 = .4$. So we simply use 5 "compoundings" at this rate to see that the amount in the account will be $A = 2000 \cdot (1.4)^5 = 10,756.48$.

19. A cubic polynomial $p(x)$ with leading coefficient 1 has three zeros, $x = 1$, $x = -1$, and $x = 3$. What is the value $p(2)$?

(A) -3 **(B)** -1 **(C)** 1 **(D)** 2 **(E)** 3

Solution: A. The polynomial is $p(x) = a(x-1)(x+1)(x-3)$, and since the leading coefficient of p is 1, it follows that $a = 1$. Thus $p(2) = 1 \cdot 3 \cdot (-1) = -3$.

20. Which of the following numbers is closest to the sum of the roots of the given equation?

$$|2x - 5| = x^2 - 2x - 2$$

(A) -1.25 **(B)** -0.75 **(C)** 0.35 **(D)** 1.15 **(E)** 2.12

Solution: C. Condition on the value of x. If $x \geq 2.5$ then the equation becomes $2x - 5 = x^2 - 2x - 2$ which has one solution larger than 2.5, namely $x = 3$. If $x < 2.5$ then the equation becomes $-2x + 5 = x^2 - 2x - 2$ and this leads to $x^2 = 7$. Note that $\sqrt{7} > 2.5$. Thus the two roots are $x = 3$ and $x = -\sqrt{7}$ and their sum is very close to 0.35.

21. Let p denote the smallest prime number greater than 200 for which there are positive integers a and b satisfying

$$a^2 + b^2 = p.$$

What is $a + b$?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Solution: B. Note that $a^2 + b^2$ is even if both a and b have the same parity. Since $a^2 + b^2$ is odd, one of a and b is odd and the other is even. Suppose a is odd. Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1$ is one bigger than a multiple of 4. Also $b^2 = (2l)^2 = 4l^2$ is a multiple of 4.. Thus $a^2 + b^2$ is one bigger than a multiple of 4. Checking the table at the back, we see that the next prime of the form $4n + 1$ is 229. Finally, $229 = 2^2 + 15^2$, so $a = 2$ and $b = 15$ and $a + b = 17$.

22. Two rational numbers r and s are given. The numbers $r + s, r - s, rs$, and s/r are computed and arranged in order by value to get the list $1/3, 3/4, 4/3, 7/3$. What is the sum of the squares of r and s?

(A) 9/25 **(B)** 4/9 **(C)** 9/4 **(D)** 25/9 **(E)** 6

Solution: D. Because both rs and $r + s$ are positive, both r and s are positive. Since $r - s > 0$, it follows that $s < r$ and thus $s/r < 1$. Thus either $s/r = 1/3$ or $s/r = 3/4$. If $s/r = 1/3$ then $3s = r$ and $r + s = 4s$ and $r - s = 2s$. In case $s/r = 3/4$, then $4s = 3r$ and $r + s = r + 3r/4 = 7r/4$ and $r - s = r - 3r/4 = r/4$ so $r + s = 7(r - s)$. It follows that $r + s = 7/3$ and $r - s = 1/3$. From this it follows that $r = 4/3$, $s = 1$ and $r^2 + s^2 = 25/9$.

An alternate solution starts the same way, with rs and $r + s$ positive, so both r and s are positive. Thus $r > s$ and $s/r < 1$. But then $r + s > r - s$, so $r + s$ must be greater than 1 since only two of the numbers are less than 1. Consider $(r + s)2 = r^2 + 2rs + s^2$. If $r + s = 4/3$, then s/r and $r - s$ are the two numbers less than one leaving $rs = 7/3$. But then the left side is $(r + s)^2 = 16/9$ and the right side is $r^2 + 2rs + s^2 = 14/3 + r^2 + s^2 > 14/3 > 16/9$, a contradiction. So we must have $r + s = 7/3$ and $49/9 = (r + s)^2 = r^2 + 2rs + s^2 \leq r^2 + s^2 + 8/3$. Thus $25/9 \leq r^2 + s^2 < 49/9 < 6$ and therefore the only choice is $25/9$.

23. Suppose x, y, z, and w are real numbers satisfying $x/y = 4/7$, $y/z = 14/3$, and $z/w = 3/11$. When $(x + y + z)/w$ is written in the form m/n where m and n are positive integers with no common divisors bigger than 1, what is $m + n$?

    (A) 20 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 37

    Solution: D. Calculate x/w, y/w and compute the sum. For example, $x/w = 4/7 \cdot 14/3 \cdot 3/11 = 8/11$ and $y/w = 14/3 \cdot 3/11 = 14/11$, so $(x + y + z)/w = (8 + 14 + 3)/11 = 25/11$. Therefore $m + n = 25 + 11 = 36$.

24. An integer is randomly selected from the set $\{100, 101, \ldots, 999\}$. What is the probability that the sum of its digits is the same as the product of its digits?

    (A) 0 **(B)** 1/900 **(C)** 1/300 **(D)** 1/150 **(E)** 1/100

    Solution: D. There are just six such numbers, $123, 132, 213, 231, 312, 321$, so the probability is $6/900 = 1/150$.

25. Four positive integers a, b, c and d satisfy $abcd = 10!$. What is the smallest possible sum $a + b + c + d$?

    (A) 170 **(B)** 175 **(C)** 178 **(D)** 183 **(E)** 185

    Solution: B. The sum $a + b + c + d$ is smallest when all four numbers are equal. Then $a^4 = 10!$ and $a \approx 43.64$. Thus $a + b + c + d \approx 174.58$ Since all four are integers, $a + b + c + d \geq 175$. This optimum is obtained for $a = 40, b = 42, c = 45$, and $d = 48$.

26. What is the units digit of the (decimal representations of the) product of all the prime numbers less than 500?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 5 **(E)** 6

    Solution: A. The product includes the numbers 2 and 5, so it must be a multiple of 10.

27. In a 10-team baseball league, each team plays each of the other teams 18 times. No game ends in a tie, and, at the end of the season, each team is the same positive number of games ahead of the next best team. What is the greatest number of games that the last place team could have won?

(A) 27 **(B)** 36 **(C)** 54 **(D)** 72 **(E)** 90

Solution: D. The number of games played is $\frac{18(9)10}{2} = 810$. If n is the number of wins of the last-place team, and d is the common difference of wins between successive teams, then $n + (n + d) + (n + 2d) + \cdots + (n + 9d) = 10n + 45d = 810$ so $2n + 9d = 162$. Now n is the maximum when d is a minimum (but not zero, because there are no ties). The smallest integral value of d for which n is integral is $d = 2$. Thus $n = 72$.

List of Primes between 1 and 500:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499					