UNC Charlotte 2003 Comprehensive

1. For what value of k are the lines $2x+3y=4k$ and $x-2ky=7$ perpendicular?

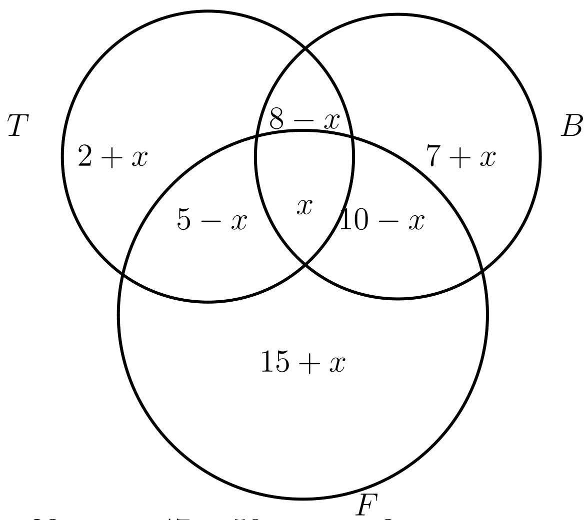
   (A) $-3/4$ **(B)** $1/6$ **(C)** $1/3$ **(D)** $1/2$ **(E)** $2/3$

   (C) The slope of the first line is $-2/3$ so the slope of the second, which is $1/2k$, must be $3/2$. Hence, $k=1/3$.

2. Conard High School has 50 students who play on the baseball, football, and tennis teams. Some students play more than one sport. If 15 play tennis, 25 play baseball, 30 play football, 8 play tennis and baseball, 5 play tennis and football, and 10 play baseball and football, determine how many students play all three sports.

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

   (B) Let TBF represent the number of students that play all three sports, let B be the number of students that play baseball, let F be the number of students that play football, and let T be the number of students that play tennis. Let TB, BF, and TF represent the number of students who play the appropriate combinations of the individual sports. Then a Venn diagram shows that the total number of students, 50 equals $T+B+F-TB-TF-BF+TBF$. Therefore $50=15+25+30-8-5-10+TBF$ which implies that $TBF=3$. Alternatively, label each region of the Venn diagram as shown.



   Then $4x+24-3x+23=x+47=50$, so $x=3$

3. Which of the following is an equation of the line tangent to the circle $x^2+y^2=2$ at the point $(1,1)$?

   (A) $x=1$ **(B)** $y=1$ **(C)** $x+y=2$ **(D)** $x-y=0$ **(E)** $x+y=0$

   (C) The center of our circle is $(0,0)$ and the line connecting the center of the circle with $(1,1)$ has slope 1. Thus the tangent line must have slope -1 and pass through $(1,1)$. The point-slope form of the equation of the tangent line is $y-1=-(x-1)$, which may be rearranged into $x+y=2$.

4. Let r and s be the two solutions to the equation $x^2-3x+1=0$. Find r^3+s^3.

   (A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 24

6. A recent poll showed that nearly 30% of European school children think that

$$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}.$$

This is wrong, of course. Is it possible that $\frac{1}{a} + \frac{1}{b} = \frac{2}{a+b}$ for some real numbers a and b?

(A) Yes, but only if $a + b = 1$ **(B)** Yes, but only if $a + b = 2$

(C) Yes, but only if $a^2 + b^2 = 1$ **(D)** Yes, but only if $a^2 + b^2 = 0$

(E) No, it is not possible

(E) The given equation is equivalent to $\frac{a+b}{ab} = \frac{2}{a+b}$, which is true only if $a^2 + 2ab + b^2 = 2ab$, but this can happen only when $a^2 + b^2 = 0$, which implies that $a = b = 0$, in which case the equation is nonsense.

7. How many ordered pairs (a, b) of integers satisfy $a^2 = b^3 + 1$?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** at least 5

   (E) Since $b^3 = a^2 - 1 = (a-1)(a+1)$, $a+1$ could be the square of $a-1$. This leads to the solution $a = 3$ and $b = 2$. By symmetry, the pair $(a, b) = (-3, 2)$ also works. It could also happen that one of $a - 1$ and $a + 1$ is zero. In these cases, we get $(1, 0), (-1, 0)$, and $(0, -1)$. In fact there are only five solutions, but the proof is well beyond the scope of this contest. For more information about this contact `hbreiter@email.uncc.edu`

8. One leg of a right triangle is two meters longer than twice the length of the other leg. The hypotenuse is four meters less than the sum of the lengths of both legs. What is the perimeter of the triangle, in meters?

   (A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 50

   (C) The three sides have lengths x, $2x+2$, and $3x-2$, and these numbers satisfy the Pythagorean identity. Thus, $x^2+(2x+2)^2 = x^2+4x^2+8x+4 = 9x^2-12x+4$, which is equivalent to $4x^2 - 20x = 0$, so the solutions are $x = 0$ and $x = 5$. Therefore the perimeter is $5 + 12 + 13 = 30$.

9. Fifteen numbers are picked from the 21-element set $\{1, 2, 3, \ldots 20, 21\}$. What is the probability that at least three of those numbers are consecutive?

   (A) 0 **(B)** 0.5 **(C)** 0.9 **(D)** 0.99 **(E)** 1

   (E) Consider the 'baskets' $\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \ldots, \{19, 20, 21\}$. Put each of the 15 numbers into the set it helps to name. There are 7 subsets and 15 numbers so, by the Pigeonhole Principle, at least one of the baskets must have three members.

10. Margaret and Cyprian both have some nickels, dimes and quarters, at least one of each type and a different number of each type. Margaret has the same number of quarters as Cyprian has dimes, and she has the same number of dimes as Cyprian has nickels. She also has the same number of nickels as Cyprian has quarters. The value of their coins is the same. What is the smallest possible total value of Margaret's coins in cents?

    (A) 85 **(B)** 95 **(C)** 105 **(D)** 125 **(E)** 135

    (A) Let K denote this least number of cents Margaret could have and let n, d, and q denote the number of nickels, dimes, and quarters respectively Margaret has. Then $5n + 10d + 25q = K$ and $25n + 10q + 5d = K$. Subtracting one equation from the other yields $d = 4n - 3q$. We want to minimize q, so we try $q = 1$. Then $n = 1$ is not allowed so we try $n = 2$ and we get $d = 5$, so the value of K is 85. The other possible values can be easily eliminated.

11. Let S denote the set $\{(-2,-2),(2,-2),(-2,2),(2,2)\}$. How many circles of radius 3 in the plane have exactly two points of S on them?

    (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 16

    (D) Each pair of points of S belongs to two circles of radius 3. There are no three points on the same circle of radius 3. There are six pairs of points, so there are 12 circles.

12. The sum of the cubes of ten consecutive integers is 405. What is the sum of the ten integers?

    (A) 10 **(B)** 15 **(C)** 20 **(D)** 23 **(E)** 24

    (B) The integers cannot all be positive because the sum of the cubes would be greater than 405. Testing a few cubes, we see that $4^3+5^3+6^3 = 64+125+216 = 405$ so the ten integers must be $-3,-2,-1,0,1,2,3,4,5,6$, and their sum is 15.

13. Let M and N denote the two integers that are respectively twice and three times the sum of their digits. What is $M + N$?

    (A) 27 **(B)** 36 **(C)** 45 **(D)** 54 **(E)** 60

    (C) Note that both M and N must be two-digit numbers. Let $M = 10a + b$. Then $2(a + b) = 10a + b$ and it follows that $8a = b$ and $M = 18$. Similarly, we can show that $N = 27$, so their sum is 45.

14. How many positive integer triples (x, y, z) satisfy $\frac{1}{x} + \frac{y}{z} = \frac{13}{21}$, where $x < 7$ and where y and z have no common divisors?

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

    (D) There are just 5 integer values of x to try and each one works, so we can find y and z by reducing the fraction $\frac{y}{z} = \frac{13}{21} - \frac{1}{x}$. The five solutions are $(x, y, z) = (2, 5, 42); (3, 2, 7); (4, 31, 84); (5, 44, 105)$; and $(6, 19, 42)$.

15. Let $D(a, b, c)$ denote the number of multiples of a that are less than c and greater than b. For example, $D(2, 3, 8) = 2$ because there are two multiples of 2 between 3 and 8. What is $D(9^3, 9^4, 9^6)$?

    (A) 71 **(B)** 719 **(C)** 720 **(D)** 7200 **(E)** 72000

    (B) The multiples of 9^3 between 9^4 and 9^6 are $10(9^3), 11(9^3), 12(9^3), \ldots, (9^3 - 1)(9^3)$. The sequence $10, 11, 12, \ldots, 9^3 - 1$ has 719 members. Alternatively, use the fact that for integers $n \leq m$ there are $m - n + 1$ integers in the range $n \leq m$. So we have $(9^3 - 1) - 10 + 1 = 719$, since $10 \cdot 9^3$ is the smallest multiple of 9^3 that is larger than $9^4 = 9 \cdot 9^3$ and $(9^3 - 1) \cdot 9^3$ is the largest multiple of 9^3 that is smaller than $9^6 = 9^3 \cdot 9^3$.

16. A $4 \times 4 \times 4$ wooden cube is painted on all 6 faces and then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that exactly one of the five visible faces is painted?

    (A) 5/16 **(B)** 7/16 **(C)** 15/31 **(D)** 31/64 **(E)** 1/2

    (B) There are $6 \cdot 4 = 24$ cubes with one face painted, and these show one painted face with probability 5/6. There are $12 \cdot 2 = 24$ cubes with two painted faces and these show one painted face with probability 1/3. The other 16 cubes show one painted face with probability 0. So the probability that one painted face shows is $p = 24/64 \cdot 5/6 + 24/64 \cdot 1/3 = 7/16$.

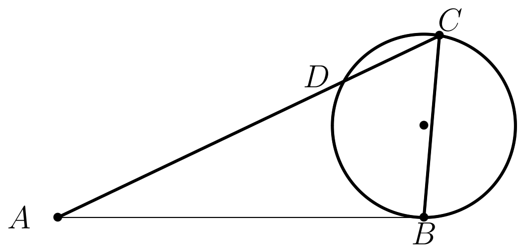
17. How many non-empty sets T of natural numbers satisfy the property: $s \in T$ implies $\frac{8}{s} \in T$. ?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

    (D) Both s and $\frac{8}{s}$ should be factors of 8 and T should contain the pair $\{s, \frac{8}{s}\}$. Thus, the possible sets are $T = \{1, 2, 4, 8\}, T = \{1, 8\},$ and $T = \{2, 4\}$.

18. An interesting fact about circles is that if a line AB is tangent to a circle at the point B and a different line through A intersects the circle in points C and D as in the diagram below, then $AB^2 = AC \cdot AD$. If $AC = 4$, $AD = 3$ and $BC = 2$, what is the measure of the angle DAB to the nearest degree? Note that the chord $\overline{BC}$ does not pass through the center of the circle.

    (A) 15 **(B)** 22.5 **(C)** 28 **(D)** 30 **(E)** 33



(D) First note that $AB^2 = AC \cdot AD = 4 \cdot 3 = 12$. Then by the law of cosines, $BC^2 = AC^2 + AB^2 - 2AC \cdot AB \cdot \cos \angle DAB$. Thus $2^2 = 4^2 + 12 - 2 \cdot 4 \cdot \sqrt{12} \cos \angle DAB$. We can solve $\cos \angle DAB = 24/(16\sqrt{3})$ to get $\angle DAB = \arccos \sqrt{3}/2 = 30.0°$.

19. What is the smallest value of the positive integer n for which

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

is at least 1? *Hint:* Rewrite each $\frac{1}{k \cdot (k+1)}$ in the from $\frac{1}{a} - \frac{1}{b}$.

(A) 10 **(B)** 100 **(C)** 1000 **(D)** 2002 **(E)** there is no such value of n

(E) Since $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k \cdot (k+1)}$, we get

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} &= \\ \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) &= 1 - \frac{1}{n+1} \end{aligned}$$

which is always less than 1.

20. Consider the number x defined by the periodic continued fraction

$$x = \frac{1}{2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \cdots}}}}.$$

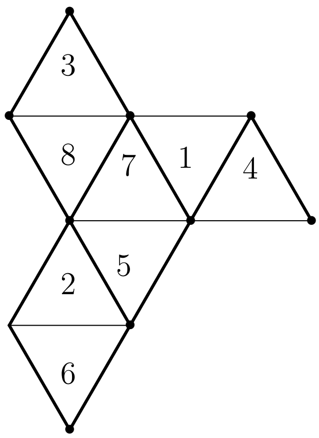
Then $x =$

(A) $\dfrac{1}{3}$ **(B)** $\dfrac{3}{7}$ **(C)** $\dfrac{-3 - \sqrt{15}}{2}$ **(D)** $\dfrac{-3 + \sqrt{15}}{2}$ **(E)** None of these

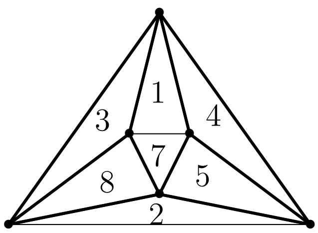
(D) Note that x is embedded in its defining fraction, and that $x = \frac{1}{2 + \frac{1}{3 + x}}$. This is equivalent to the quadratic $2x^2 + 6x - 3 = 0$ which has two solutions, $\frac{-3 - \sqrt{15}}{2}$ and $\frac{-3 + \sqrt{15}}{2}$, the first of which is extraneous.

21. An octahedral net is a collection of adjoining triangles that can be folded into a regular octahedron. When the net below is folded to form an octahedron, what is the sum of the numbers on the faces adjacent to one marked with a 3?

    (A) 13 **(B)** 15 **(C)** 17 **(D)** 18 **(E)** 19



(B) We can build the net in a different way showing all the faces except the one with the 6, which is at the back in the figure below.



Note first that the face with the 7 is adjacent to those with numbers 1, 5, and 8. Filling in those four values in the center triangles, we can then see that the face with the 4 must be adjacent to those with the 1 and the 5 and similarly with the faces marked 3 and 2. This shows that the face marked 3 is adjacent to those marked with 1, 8 and 6 so the total value is 15.

22. The stairs leading up to to the entrance of Joe's house have 5 steps. Playful Joe is able to go up 1 step or 2 steps at a time. How many ways are there for him to reach the top of the stairs? In other words, how many ways are there to write 5 as a sum of 1's and 2's if the order of the summands matters?

    (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 10

    (D) One way is to list all possibilities: $1+2+2=5$, $1+1+1+2=5$, $2+1+2=5$, $1+1+2+1=5$, $2+2+1=5$, $1+2+1+1=5$, $1+1+1+1+1=5$, $2+1+1+1=5$.

    Alternatively, we can denote by F_n the number of ways Joe can reach the top of the stairs if there are n steps. The first leap is 1 step or 2 steps, and there are F_{n-1}, respectively F_{n-2} ways to complete the procedure. This observation proves the recursion formula $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 2$. From here we see that F_n is the n-th Fibonacci number, and so $F_5 = 8$.

23. How many of the 1024 integers in the set $1024, 1025, 1026, \ldots, 2047$ have more 1's than 0's in their binary representation?

    (A) 252 **(B)** 512 **(C)** 638 **(D)** 768 **(E)** 772

    (C) Each number in the set has an 11-digit binary representation that starts with 1. Therefore it takes five more 1's in order to have more 1's than 0's. Hence we can build such a number by selecting five or more places to put the 1's. Thus, the number is $\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10} = \frac{1}{2}\left(2^{10}-\binom{10}{5}\right)+\binom{10}{5} = 386+252 = 638$.

24. Suppose x and y are integers satisfying both $x^2 + y = 62$ and $y^2 + x = 176$. What is $x + y$?

    (A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

    (A) Subtract the former from the latter to get $y^2 - x^2 - (y - x) = 114$, which factors, so that we can write $(y - x)(y + x - 1) = 2 \cdot 3 \cdot 19$. Trying various combinations yields $y - x = 6$ from which it follows that $x = 7$ and $y = 13$, so their sum is 20. Alternatively, replace y with $62 - x^2$ in the second equation and use the graphing calculator. OR, use $y - x = 6$ to conclude directly that $y + x - 1 = 19$ from the fact that $(y - x)(y + x - 1) = 6 \cdot 19$, and therefore $y + x = 20$.

25. Find all solutions of the equation $\log_2 x + \log_2(x+2) = 3$.

 (A) There is one solution x, and $x \geq 3$.

 (B) There is one solution x, and $1 < x < 3$.

 (C) There are two positive solutions.

 (D) There is one positive solution and one negative solution.

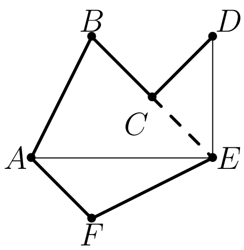
 (E) There is are no solutions.

 (B) Since $\log_2 x(x+2) = 3$, we see that $x(x+2) = 2^3$. Solving this equation, we obtain $x = 2$ and $x = -4$. But the negative solution is not valid since $\log_2(-4)$ is not defined.

26. Find the area of the polygon $ABCDEF$ whose vertices are $A = (-1,0), B = (0,2), C = (1,1), D = (2,2), E = (2,0)$, and $F = (0,-1)$.

 (A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** 6

 (D) Looking at the diagram, the area of $\triangle ABE$ is 3, the area of $\triangle AEF$ is 1.5, and the area of $\triangle CDE$ is 1. The total area is therefore 5.5.



27. Let $f(a,b,c) = \frac{a+b}{c} + \frac{b+c}{a} + \frac{a+c}{b}$. An integer u exists such that $f(u-1, u, 2u) = 8$. What is the value of $f(u^2-3, 2u-3, 2u-4)$?

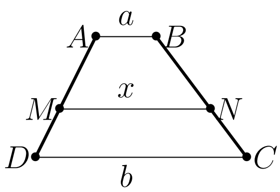
 (A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

 (E) First solve $\frac{2u-1}{2u} + \frac{3u}{u-1} + \frac{3u-1}{u} = 8$. The equation is equivalent to $14u^2 - 11u + 3 = 8(2u)(u-1)$ which leads to $u = 3$. Thus we need to compute $f(6,3,2)$ which is 8. We can disregard the case $u = -1/2$

28. Consider the trapezoid $ABCD$ (see the diagram). Suppose $\overline{MN}$ is parallel to $\overline{DC}$, $AB = a, DC = b$, and $MN = x$. If the area of $ABNM$ is half the area of $ABCD$, express x as a function of a and b.

(A) $x = \dfrac{a+b}{2}$ **(B)** $x = \dfrac{b-a}{2}$ **(C)** $x = \dfrac{3b-a}{2}$

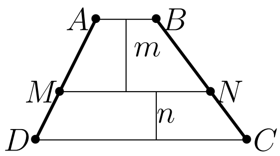
(D) $x = \sqrt{a^2+b^2}$ **(E)** $x = \dfrac{\sqrt{a^2+b^2}}{2}$



(E) Let m and n be lengths of the altitudes as shown in the diagram below. Since the area of $ABMN$ is half the area of $ABCD$, we have $(m+n)(\frac{a+b}{2}) = 2m(\frac{x+a}{2})$, which is equivalent to

$$(1 + \frac{n}{m})(b+a) = 2(x+a) \qquad \aleph$$

Furthermore, equating the areas of $ABNM$ and $MNCD$ we have $m(\frac{a+x}{2}) = n(\frac{b+x}{2})$, so that $\frac{n}{m} = \frac{a+x}{b+x}$. Substituting into $\aleph$, we have $(1 + \frac{a+x}{b+x})(b+a) = 2(x+a)$. After multiplying both sides by $b+x$ and simplifying, we see that $a^2 + b^2 = 2x^2$.



Alternatively, we can use the similarity of the triangles of the figure below.