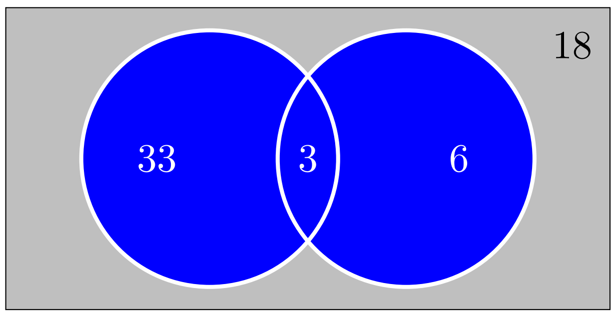
Solutions

1. A survey of 60 graduate and undegraduate students attending a university found that 9 were living off campus and 36 were undergraduates. Also, 3 were undergraduates living off campus. How many students are graduate students living on campus?

   (A) 9 (B) 18 (C) 19 (D) 20 (E) 21

   Answer: (B)

   Solution: An easy way to solve this problem is to draw a diagram.



   The students that are graduate and living on campus is the whole area outside the two circles. That is, 18 students.

2. How many real solutions does the equation $\sqrt{x} + 2x - 1 = 13$ have?

   (A) no solutions (B) one (C) two (D) three (E) more than three solutions

   Answer: (B)

   Solution: Answer B. Each of the functions $\sqrt{x}$ and $2x - 1$ are increasing functions on $[0, \infty)$. Therefore, so too is the function $f(x) = \sqrt{x} + 2x - 1$. Since it has range $[-1, \infty)$, the given equation has a unique solution.

3. The fifth term of an arithmetic sequence is 49 and the difference between successive terms of the sequence is 8. What is the sum of the first 20 terms of the sequence?

   (A) 177 (B) 500 (C) 1689 (D) 1860 (E) 2050

   Answer: (D)

   Solution: (D) Let $a_k = b \cdot k + c$ be the k-th term in the sequence. The first sentence gives $b = 8$ and $a_5 = 8 \cdot 5 + c = 49$. Hence $c = 9$.
   Then $\sum_{k=1}^{20} a_k = 8\sum_{k=1}^{20} k + \sum_{k=1}^{20} 9 = 8 \cdot (20 \cdot 21/2) + 20 \cdot 9 = 1860$.

4. Let $x = (2^{\log_2(3)})^{\frac{1}{\log_4(3)}}$. Then $x =$

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 6

   Answer: (D)

   Solution: The definition of the logarithms implies that $a^{\log_a(b)} = b$ and $\log_a(b) = 1/\log_b(a)$. Therefore, $2^{\log_2(3)} = 3$ and $x = 3^{\log_3(4)} = 4$. The answer is D.

5. What is the area of the smallest disc that contains the isosceles right triangle with leg length 2?

   (A) 2π (B) $\sqrt{2}\pi$ (C) $2\sqrt{2}\pi$ (D) 4π (E) $\frac{\sqrt{2}}{2}\pi$

   Answer: (A)

   Solution: The smallest such disc would have the hypotenuse of the triangle as a diameter, so its radius is $\sqrt{2}$.

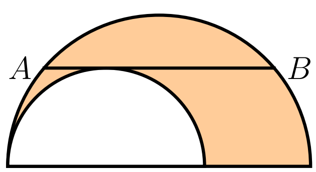
6. Three friends make a pizza together. The first friend covers 80% of the pizza with peppers. The second friend covers 75% of the pizza with onions. The third friend covers 70% of the pizza with artichokes. How much of the pizza is guaranteed to have all three toppings?

   (A) 0% (B) 20% (C) 25% (D) 30% (E) 70%

   Answer: (C)

   Solution: (C) To minimize the fraction of the pizza having all three toppings, is to maximize its fraction lacking at least one of them. The fractions of the pizza lacking pepper, onions or artichokes, are 20%, 25%, and 30%, respectively. The total fraction of those pieces is maximized when they are disjoint, in which case the total is $20\% + 25\% + 30\% = 75\%$.

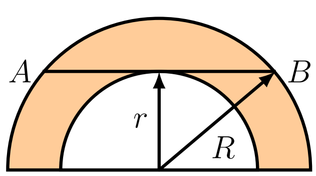
7. The chord AB is parallel to the diameters of both semi-disks. Its length $|AB| = 24$. What is the area of the shaded figure?



   (A) 48π (B) 60π (C) 72π (D) 84π (E) 96π

   Answer: (C)

   Solution: The area S of the shaded figure is the same as that shown below. It equals $S = \frac{1}{2}\pi R^2 - \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(R^2 - r^2\right) = \frac{\pi}{2} \cdot 12^2 = 72\pi$



8. A three-digit number is drawn uniformly at random. What is the probability that the sum of its digits is 3?

(A) $\frac{5}{890}$ (B) $\frac{1}{150}$ (C) $\frac{1}{180}$ (D) $\frac{1}{225}$ (E) $\frac{6}{899}$

Answer: (B)

Solution:

(B) There are 900 three-digit numbers. There are 6 three-digit numbers whose digits sum to 3: $111, 102, 120, 201, 210$, and 300. Thus, the probability is $6/900 = 1/150$.

9. Two cards are selected randomly and simultaneously from a set of five cards numbered 3, 6, 11, 15, and 17. What is the probability that both cards selected are prime numbered cards?

(A) 1/5 (B) 2/5 (C) 1/10 (D) 3/10 (E) 3/5

Answer: (D)

Solution: Three cards have prime numbers, so the number of outcomes resulting in two prime numbers is $\binom{3}{2} = 3$. The total number of possible outcomes is $\binom{5}{2} = 10$. Therefore, the probability sought for equals 3/10.

10. The sides of a right triangle have lengths a, $2a+2d$ and $2a+3d$, with a and d both positive. The ratio of a to d is:

(A) $5:1$ (B) $81:6$ (C) $4:1$ (D) $1:5$ (E) $2:3$

Answer: (A)

Solution: For the right triangle, the longest side is the hypotenuse, therefore

$$\begin{aligned}
a^2 + (2a+2d)^2 &= (2a+3d)^2 \\
a^2 + (4a^2 + 8ad + 4d^2) &= 4a^2 + 12ad + 9d^2 \\
a^2 - 4ad - 5d^2 &= 0 \\
(a-5d)(a+d) &= 0 \\
a = 5d \quad &\text{or} \quad a = -d \qquad (1)\\
&\text{(dropped since } a \text{ and } d > 0).
\end{aligned}$$

$a : d = 5d : d = 5 : 1.$

11. A cube of edge length equal to R is placed inside a sphere of radus R, so that the center of the sphere coincides with the center of the cube. The planes containing faces of the cube cut the sphere into several pieces. Into how many pieces do the planes cut the sphere?

    (A) 15 (B) 26 (C) 12 (D) 10 (E) 16

    Answer: (B)

    Solution: No point of the cube is on the sphere so there is no change to the number of pieces if we deform the sphere into a cube with edge lengths $3R$ so that when it is viewed as a $3x3$ block of cubes of edge length R, the original cube is the center cube. Each outer block of the $3x3$ arrangement contains a single piece that is uncut by the planes of the faces of the inner (middle) cube. Since these account for the entire outer shell, this accounts for all the pieces. There $3^3 - 1$ many such outer blocks.

12. A point (x, y) is selected at random uniformly in the square $Q = [-2, 2] \times [-2, 2]$. Let P be the probability that $|y| \leq |x|$ and, at the same time, $x^2 + y^2 \leq 2$. Which of the following intervals contains P?

    (A) [0, 0.1] (B) (0.1, 0.2] (C) (0.2, 0.3] (D) (0.3, 0.4] (E) (0.4, 1]

    Answer: (B)

    Solution: The event whose probability P is sought for occurs when the random point (x, y) is in the region G that is the union of two sectors of the disk of radius $\sqrt{2}$ centered at the origin: the first sector consists of those points of the disk whose polar angle θ satisfies the inequality $-\pi/4 \leq \theta \leq \pi/4$, while the second sector consists of those points of the disk for which $3\pi/4 \leq \theta \leq 5\pi/4$ (in addition, each sector contains the origin). The total area of the two sectors is

$$\frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \cdot (\sqrt{2})^2 = \pi.$$

    Therefore,

$$P = \frac{\pi}{4^2} = \frac{\pi}{16}.$$

    Since $1.6 < \pi < 3.2$, we have $0.1 = \frac{1.6}{16} < \frac{\pi}{16} < 0.2$.

13. Let p be the greatest prime factor of 9991. The sum s of the digits of p is equal to

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

Answer: (A)

Solution: Observe that $9991 = 10000 - 9 = 100^2 - 3^2 = (100 - 3) \cdot (100 + 3) = 97 \cdot 103$. Both numbers are prime. Hence, $s = 1 + 0 + 3 = 4$.

14. Suppose that $f(x) = \dfrac{4^x}{4^x + 2}$. Find the value of $S = f(\frac{1}{14}) + f(\frac{2}{14}) + f(\frac{3}{14}) + \ldots + f(\frac{13}{14})$.

(A) 4.5 (B) 5.5 (C) 6.5 (D) 7.5 (E) 8.5

Answer: (C)

Solution: Observe that $f(x) + f(1 - x) = \dfrac{4^x}{4^x + 2} + \dfrac{4^{1-x}}{4^{1-x} + 2} = \dfrac{4^x}{4^x + 2} + \dfrac{2}{2 + 4^x} = 1.$

Therefore,

$$\begin{aligned} 2S = {} & f\left(\frac{1}{14}\right) + f\left(\frac{2}{14}\right) + \ldots + f\left(\frac{13}{14}\right) \\ & + f\left(\frac{13}{14}\right) + f\left(\frac{12}{14}\right) + \ldots + f\left(\frac{1}{14}\right) = 13. \end{aligned}$$

15. A collection of 120 wooden cubes is used to build a $4 \times 5 \times 6$ block of cubes. Five of the six faces are painted red. Then a cube is randomly selected from the 120 cubes and then rolled like a die. Which of the following could be the probability that a red face comes up.

(A) 1/6 (B) 31/180 (C) 8/47 (D) 1/5 (E) 1/4

Answer: (B)

Solution: B. The number of painted faces must be the surface area minus the area, S, of one of the faces, $20, 24$ or 30. So the number of painted faces is $2(20 + 24 + 30) - S$. That is $118, 124$, or 128. On the other hand the total number of faces is $6 \cdot 120 = 720$, so the three possible probabilities are $\frac{118}{720} = \frac{59}{360}$, $\frac{124}{720} = \frac{31}{180}$, or $\frac{128}{720} = \frac{8}{45}$. Only the second of these is listed.

16. Into how many regions does the set $S = \{(x, y)|\ x^2 - y^2 + 4x + 6y - 5 = 0\}$ divide the plane?

    (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

    Answer: (E)

    Solution: E. We have $S = \{(x, y)|\ (x + 2)^2 - (y - 3)^2 = 0\}$, or, equivalently,

    $$S = \{(x, y)|\ (x - y + 5)(x + y - 1) = 0\} = \{(x, y)|\ y = x + 5 \text{ or } y = -x + 1\},$$

    so that S is the union of two intersecting lines.

17. Astronomers, while observing stars $F,\ E,\ D,\ L,\ K,\ H,\ G$, found the following straight line distances (measured in light-years):
    $|FE| = 140,\ |ED| = 24;\ |DL| = 266;\ |LK| = 12;\ |KH| = 16;\ |HG| = 5;\ |GF| = 69.$
    Find the distance $|EH|$.

    (A) 108 (B) 112 (C) 206 (D) 214 (E) 228

    Answer: (D)

    Solution: We have $266 = 24 + 140 + 69 + 5 + 16 + 12$, that is, $|DL| = |DE| + |EF| + |FG| + |GH| + |HK| + |KL|$, so that all the stars are on the same line, in this order: $D,\ E,\ F,\ G,\ H,\ K,\ L$. Therefore, the distance between E and H equals $|EF| + |FG| + |GH| = 140 + 69 + 5 = 214$.

18. Let $\triangle ABC$ be an equilateral triangle with sides of length 1. Let P be any point inside the triangle. What is the sum of the distances of P from each of the three sides?

    (A) 1 (B) 1/2 (C) $\sqrt{3}$ (D) $\sqrt{3}/2$ (E) $\sqrt{3}/4$

    Answer: (D)

    Solution: Answer: (D)

    Let d_a, d_b and d_c be the distances between P and the sides BC, AC and AB, respectively. The area of the triangle $\triangle ABP$ is $d_c/2$, similarly the area of $\triangle ACP$ is $d_b/2$ and the area of the triangle $\triangle BCP$ is $d_a/2$. The sum of these three areas is the area of the triangle $\triangle ABC$, which is $\sqrt{3}/4$. Hence we must have $d_a + d_b + d_c = \sqrt{3}/2$

19. At a game show, after hearing the last question, you have the option to keep your current win and go home, or to answer the question. If you answer the last question correctly, the amount you won gets doubled. If you give an incorrect answer, you get to take home only half of your current win. The probability that the answer you have in mind is correct is p. What is the smallest p that would make your expected win, if you answer the last question, at least as large as your current win?

(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) 3/4

Answer: (B)

Solution: Answer: (B)

Suppose your current win is A. If you go on, your win is $2A$ with probability p and $A/2$ with probability $1-p$. Your expected win is $2p \cdot A + \frac{1-p}{2} \cdot A$. It is worth continuing only if this is at least A, that is

$$2p + \frac{1-p}{2} \geq 1.$$

The solution of this inequality is $p \geq 1/3$.

20. A sequence of numbers starts with the number 2017^{2017} and the next number is the current number minus the sum of its digits. What is the last number in the sequence that is not 0?

(A) 12 (B) 8 (C) 9 (D) 1 (E) 5

Answer: (C)

Solution: For any number $n \geq 1$, $n = \Sigma_{i=0}^{k} a_i 10^i$, the number n minus the sum of its digits, i.e. $\Sigma_{i=0}^{k} a_i 10^i - \Sigma_{i=0}^{k} a_i$ is equal to $\Sigma_{i=1}^{k} a_i (10^i - 1)$ and is therefore a multiple of 9.