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PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2009. Solutions identified as by students are given preference.*

Problems for Solution.

1196. *Proposed by Sam Vandervelde, St. Lawrence University, Canton, NY.*

Let $\mathbb{Q}^* = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\ a \neq 0,\ b > 0, \text{ and } (a,b) = 1\}$. In other words, $\mathbb{Q}^*$ is the set of all nonzero rational numbers written in lowest terms. Find, with proof, the value of

$$\sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^2}.$$

1197. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.*

The Jacobsthal numbers, J_n, are defined recursively by $J_0 = 0$, $J_1 = 1$ and $J_n = J_{n-1} + 2J_{n-2}$ for all $n \geq 2$, whereas the Jacobsthal-Lucas numbers, j_n, are defined recursively by $j_0 = 2$, $j_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$ for all $n \geq 2$.

(a) Show

$$\sum_{k=0}^{n} J_k = J_{n+1} - \frac{1 + (-1)^n}{2} \quad \text{and} \quad \sum_{k=0}^{n} j_k = j_{n+1} + \frac{3(-1)^n - 1}{2}.$$

(b) For $n \geq 1$, show that

$$j_n^{-1} \cdot \left(\sum_{k=0}^{2n-1} J_k \right) \quad \text{and} \quad J_n^{-1} \left(\sum_{k=0}^{2n-1} j_k \right)$$

are integers.

1198. *Proposed by Arthur L. Holshouser, Charlotte, NC.*

Two functions $g : \mathbf{C} \cup \{\infty\} \rightarrow \mathbf{C} \cup \{\infty\}, \overline{g} : \mathbf{C} \cup \{\infty\} \rightarrow \mathbf{C} \cup \{\infty\}$ (where $\mathbf{C}$ is the complex numbers) are similar (written $g \sim \overline{g}$) if there exists a bijection $f : \mathbf{C} \cup \{\infty\} \rightarrow \mathbf{C} \cup \{\infty\}$ such that $g = f \circ \overline{g} \circ f^{-1}$ where $\circ$ is the composition of functions, and f^{-1} is the inverse function. Prove that

$$g(x) = \tan\left(2 \tan^{-1} x\right) = \frac{2x}{1 - x^2}$$

and $\overline{g}(x) = x^2$ are similar by explicitly computing a bijection $f : \mathbf{C} \cup \{\infty\} \rightarrow \mathbf{C} \cup \{\infty\}$ of the form $f(x) = \frac{ax+b}{x+d}, a, b, d \in \mathbf{C}$.

1199. *Proposed by H. A. ShahAli, Tehran, IRAN.*

Given integers k and m with $1 < k < m$; also given m vectors of a finite dimensional vector space such that sum of every k vectors is equal to k times of one of the vectors. Prove that all of the vectors are equal.

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1200. *Proposed by Peter A. Lindstrom, Batavia, NY*

For $|x| < 1$, define

$$S(x) = \sum_{n=1}^{\infty} (2(n-1)+1)x^{n-1}.$$

Find a closed form representation for $S(x)$.

1201. *Proposed by Robert Gebhardt, Hopatcong, NJ.*

Find the family of straight lines in the XY plane for which the envelope is the simple closed curve $x^4 + y^4 = 1$.

1202. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

Let $p \geq 1$ be a natural number. Prove that

$$a) \qquad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{n-1} x^{pk} - \frac{1}{1-x^p} \right) = \frac{\ln(1-x^p)}{1-x^p}, \quad -1 < x < 1.$$

$$b) \qquad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{n-1} (-1)^k x^{pk} - \frac{1}{1+x^p} \right) = \frac{\ln(1+x^p)}{1+x^p}, \quad -1 < x < 1.$$

1203. *Proposed by Arthur L. Holshouser, Charlotte, NC.*

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Find a binary operation $*$ on S such that for each three-element subset $\{x, y, z\}$ of S, exactly one of the equations $x * y = z$, $y * x = z$, $x * z = y$, $z * x = y$, $y * z = x$, $z * y = x$ holds.