Meshfree Particle Methods for Thin Plates

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Abstract

In this paper, we are concerned with meshfree particle methods for the solutions of the classical plate model. The vertical displacement of a thin plate is governed by a fourth order elliptic equation and thus the approximation functions for numerical solutions are required to have continuous partial derivatives. Hence, the conventional finite element method has difficulties to solve the fourth order problems. Meshfree methods have the advantage of constructing smooth approximation functions, however, most of the earlier works on meshfree methods for plate problems used either moving least squares method with penalty method or coupling FEM with meshfree method to deal with essential boundary conditions. In this paper, by using generalized product partition of unity, introduced by Oh et al, we introduce meshfree particle methods in which approximation functions have high order polynomial reproducing property and the Kronecker delta property. We also prove error estimates for the proposed meshfree methods. Moreover, to demonstrate the effectiveness of our method, results of the proposed method are compared with existing results for various shapes of plates with variety of boundary conditions and loads.

Keywords: Meshfree methods, generalized product partition of unity; partition of unity function with flat-top; reproducing polynomial particle shape functions; Kirchhoff plate model.

1 Introduction

A large number of structural components in engineering can be classified as plates. Typical examples in civil engineering structures are floor and foundation slabs, lock-gates, thin retaining walls, bridge decks and slab bridges. Plates are indispensable in ship building, automobile, and aerospace industries.

The stress resultants of a thin plate, such as membrane forces, shear forces and moments, can be calculated through the 3-dimensional elasticity. Under certain hypotheses such as Kirchhoff

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and Reissner-Mindlin, the 3-dimensional elasticity for a plate is reduced to a two-dimensional problem. The Kirchhoff plate model obtained through the Kirchhoff hypothesis is well suited for very thin plates. A limitation of this model is that the governing equation of the displacement vector of this plate model is a fourth order differential equation. Thus, for the conventional finite element solution of this problem, the Argyris and Bell triangles, the Bogner-Fox-Schmit rectangle ([6],[7]) are suggested to construct $C^1$-continuous finite elements. These Hermite-type finite elements are difficult to implement. By introducing two additional unknown functions, the Kirchhoff hypothesis is relaxed to have the first (or the third) order shear deformation plate model. However, these plate models have locking problems and boundary layer problems. The boundary layer problem worsens as the plate gets thinner. In this paper, we are concerned with the classical plate model. Numerous papers and books have been published on the classical plate model for thin plates ([33],[37] and the papers referenced therein) that have no boundary layer problems. Moreover, instead of using $C^1$-finite elements, for the Kirchhoff plate model, non conforming finite element methods ([16],[17],[40]) and mortar finite element methods ([24]) have also been applied.

Meshless methods ([1],[3],[4],[5],[14],[18],[34],[35]) have several advantages over the conventional finite element method ([6],[7],[36]). Their flexibility and wide applicability have gained attention from scientists and engineers to this very dynamic research area ([10]-[12]). Meshless methods use flexible smooth base functions and use no mesh or use background mesh minimally. Actually, meshless methods have been referred to as meshfree methods ([1],[3],[4]), Reproducing Kernel Particle Methods(RKPM) ([13],[18],[21],[22],[23]), Reproducing Kernel Element Methods (RKEM) ([18],[19],[20]), GFEM (PUFEM)([25],[34],[35]), $h$-$p$ Cloud Method([8]) and Element Free Galerkin Method (EFGM) ([1]).

Even though these approaches are applicable in solving many difficult science and engineering problems, they are limited by their large matrix condition numbers ([34],[35]), inefficiency in handling essential boundary conditions ([2],[5]), complexity in constructing a partition of unity ([9]), lengthy numerical integration ([9]), and so on.

To alleviate difficulties encountered in meshless methods, Oh et al introduced three closed-form partition of unity (PU) that have flat-top: (i) Convolution partition of unity ([29]) for background meshes of arbitrary convex polygonal shapes of partition of a given domain; Using convolution partition of unity, Oh et al. introduced several meshless methods that are called uniform RPPM (Reproducing Polynomial Particle Method) ([30]), patchwise RPPM, adaptive RPPM, and RSPM (Reproducing Singularity Particle Method) in ([28],[29],[31],[32]). Note that RPPM is similar to RKPM ([3],[13],[18],[19],[20],[21],[22],[23]). (ii) Almost everywhere partition of unity ([27]) for background meshes on non convex domains and imposing essential boundary condition on convex as well as non convex domains. (iii) Generalized product partition of unity ([26]) for construction of non-uniformly distributed particle shape functions. Using PU functions with flat-top gives relatively small matrix condition numbers.

In this paper, for the thin plate problems, the generalized product partition of unity is applied to construct smooth local approximation functions that have the reproducing polynomial property and the Kronecker delta property.

In section 2, we briefly review the generalized product partition of unity. Definitions and terminologies that are used in this paper are also introduced. In section 3, reproducing polynomial particle methods (RPPM) for plate problems are introduced. We estimate the error
bounds of numerical solutions of the proposed RPPM in the $L^2$-norm, the 1-seminorm and the 2-seminorm, respectively. In section 4, the variational formulation of thin plate is described. In section 5, the reference shape functions that satisfy the clamped boundary conditions of thin plates are constructed. In section 6, effectiveness of the proposed meshfree method (RPPM) is demonstrated with various shapes of plates. Finally, the concluding remarks are given in section 7.

2 Closed-form-smooth-partition of unity functions with flat-top

Let $\Omega$ be a connected open subset of $\mathbb{R}^d$. We define the vector space $C^m(\Omega)$ to consist of all those functions $\phi$ which, together with all their partial derivatives $\partial^\alpha \phi (= \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \phi)$ of orders $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq m$, are continuous on $\Omega$. In the following, a function $\phi \in C^m(\Omega)$ is said to be a $C^m$-function. If $\Psi$ is a function defined on $\Omega$, we define the support of $\Psi$ as

$\text{supp} \Psi = \{ x \in \Omega | \Psi(x) \neq 0 \}.$

For an integer $k \geq 0$, we also use the usual Sobolev space denoted by $H^k(\Omega)$. For $u \in H^k(\Omega)$, the norm and the semi-norm, respectively, are

$$\|u\|_{k, \Omega} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha u|^2 \, dx \right)^{1/2},$$

$$|u|_{k, \Omega} = \left( \sum_{|\alpha| = k} \int_{\Omega} |\partial^\alpha u|^2 \, dx \right)^{1/2},$$

$$\|u\|_{k, \infty, \Omega} = \max_{|\alpha| \leq k} \{ \text{ess.sup} |\partial^\alpha u(x)| : x \in \Omega \};$$

$$|u|_{k, \infty, \Omega} = \max_{|\alpha| = k} \{ \text{ess.sup} |\partial^\alpha u(x)| : x \in \Omega \}. \quad (1)$$

A family $\{ U_k : k \in D \}$ of open subsets of $\mathbb{R}^d$ is said to be a point finite open covering of $\Omega \subseteq \mathbb{R}^d$ if there is an integer $M$ such that any $x \in \Omega$ lies in at most $M$ of the open sets $U_k$ and $\Omega \subseteq \bigcup_k U_k$.

For a point finite open covering $\{ U_k : k \in D \}$ of a domain $\Omega$, suppose there is a family $\{ \phi_k : k \in D \}$ of Lipschitz functions on $\Omega$ satisfying the following conditions:

1. For $k \in D$, $0 \leq \phi_k(x) \leq 1$, $x \in \mathbb{R}^d$.
2. The support of $\phi_k$ is contained in $U_k$, for each $k \in D$.
3. $\sum_{k \in D} \phi_k(x) = 1$ for each $x \in \Omega$.

Then $\{ \phi_k : k \in D \}$ is called a partition of unity (PU) subordinate to the covering $\{ U_k : k \in D \}$. The covering sets $\{ U_k \}$ are called patches.

By almost everywhere partition of unity, we mean $\{ \phi_k : k \in D \}$ such that the condition 3 of a partition of unity is not satisfied only at finitely many points (2D) or lines (3D) on a part of the boundary.

Let $\omega = \text{supp}(\phi)$. Then $\omega_{fl} = \{ x \in \omega : \phi(x) = 1 \}$ and $\omega_{nfl} = \{ x \in \omega : 0 < |\phi(x)| < 1 \}$ are called the flat-top part and the non flat-top part of $\omega$, respectively. The function $\phi$ is said to
be a function with flat-top if the interior of $\omega^{flt}$ is non-void. Moreover, \{\phi_k : k \in D\} is called a partition of unity with flat-top whenever it is a partition of unity and $\phi_k$ is a function with flat-top for each $k \in D$. $\omega^{flt}$ is also denoted by $Q^{flt}$ whenever $\phi$ is associated with a patch $Q$.

Notice that if $f_1, \cdots, f_n$ are linearly independent on $\omega^{flt}$ and $\phi$ is a function with flat-top, the product functions, $\phi \cdot f_1, \cdots, \phi \cdot f_n$, are also linearly independent on $\omega$. However, if $\phi$ is not a function with flat-top, the product functions, $\phi \cdot f_1, \cdots, \phi \cdot f_n$, could be linearly dependent.

The hat functions of the conventional finite element are PU functions with no flat-top and hence PUFEM using the hat functions as PU yields a large matrix condition number in general.

A weight function (or window function) is a non-negative continuous function with compact support and is denoted by $w(x)$. Consider the following conical window function: For $x \in \mathbb{R}$,

$$w(x) = \begin{cases} (1 - x^2)^{l/2} & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

where $l$ is an integer. Then $w(x)$ is a $C^{l-1}$-function. In $\mathbb{R}^d$, the weight function $w(x_1, \cdots, x_d)$ can be constructed from a one-dimensional weight function as $w(x_1, \cdots, x_d) = \prod_{i=1}^{d} w(x_i)$.

In this paper, we use the normalized window function defined by

$$w^\delta_{\phi}(x) = A w(x/\delta),$$

where $A = [(2l + 1)!/[2l+1(l!)^2]]$ ([13]) is the constant that makes $\int_{\mathbb{R}} w^\delta_{\phi}(x)dx = 1$.

Let $\Lambda$ be a finite index set and $\Omega$ denotes a bounded domain in $\mathbb{R}^d$. Let \{x_j : j \in \Lambda\} be a set of a finite number of uniformly or non-uniformly spaced points in $\mathbb{R}^d$, that are called particles.

**Definition 2.1.** Let $k$ be a non-negative integer. Then the functions $\phi_j(x)$ corresponding to the particles $x_j, j \in \Lambda$ are called the RPP (reproducing polynomial particle) shape functions with the reproducing property of order $k$ (or simply, “of reproducing order $k$”) if and only if they satisfy the following condition:

$$\sum_{j \in \Lambda} (x_j)^\alpha \phi_j(x) = x^\alpha, \text{ for } x \in \Omega \subset \mathbb{R}^d \text{ and for } 0 \leq |\alpha| \leq k. \quad (5)$$

Note that the RPP shape functions $\phi_j, j \in \Lambda$, of reproducing order $k$ can exactly interpolate polynomials of degree less than or equal to $k$.

### 2.1 One-dimensional partition of unity functions without flat-top

For any positive integer $n$, $C^{n-1}$-piecewise polynomial basic PU functions are constructed as follows: For integers $n \geq 1$, we define a piecewise polynomial function by

$$\phi^{(pp)}_{g_n}(x) = \begin{cases} \phi^{L}_{g_n}(x) := (1 + x)^n g_n(x) & \text{if } x \in [-1, 0], \\ \phi^{R}_{g_n}(x) := (1 - x)^n g_n(-x) & \text{if } x \in [0, 1], \\ 0 & \text{if } |x| \geq 1, \end{cases} \quad (6)$$
where \( g_n(x) = a_0^{(n)} + a_1^{(n)}(-x) + a_2^{(n)}(-x)^2 + \cdots + a_{n-1}^{(n)}(-x)^{n-1} \) whose coefficients are inductively constructed by the following recursion formula:

\[
a_k^{(n)} = \begin{cases} 
1 & \text{if } k = 0, \\
\sum_{j=0}^{k} a_j^{(n-1)} & \text{if } 0 < k \leq n - 2, \\
2(a_{n-2}^{(n)}) & \text{if } k = n - 1.
\end{cases}
\]

(7)

For example,

\[
g_1(x) = 1; \quad g_2(x) = 1 - 2x; \quad g_3(x) = 1 - 3x + 6x^2; \quad g_4(x) = 1 - 4x + 10x^2 - 20x^3; \quad g_5(x) = 1 - 5x + 15x^2 - 35x^3 + 70x^4.
\]

Then, \( \phi_{pp}^{(g_n)} \) has the following properties whose proofs can be found in ([29]).

\- \( \phi_{pp}^{(g_n)}(x) + \phi_{pp}^{(g_n)}(x-1) = 1 \) for all \( x \in [0, 1] \) and \( 0 \leq \phi_{pp}^{(g_n)}(x) \leq 1 \), for all \( x \in \mathbb{R} \). Hence, \{\( \phi_{pp}^{(g_n)}(x-j) \mid j \in \mathbb{Z} \)\} is a partition of unity on \( \mathbb{R} \).

\- \( \phi_{pp}^{(g_n)}(x) \) is a \( C^{n-1} \)-function.

### 2.2 Generalized two-dimensional product partition of unity with flat-top

Using the basic PU function \( \phi_{pp}^{(g_n)} \) defined by (6), we construct a \( C^{n-1} \)-PU function with flat-top whose support is \([a - \delta, b + \delta]\) with \((a + \delta) < (b - \delta)\) in a closed form as follows:

\[
\psi_{[a,b]}^{(\delta, n-1)}(x) = \begin{cases} 
\phi_{g_n}^{R} \left( \frac{x -(a+\delta)}{2\delta} \right) & \text{if } x \in [a - \delta, a + \delta], \\
1 & \text{if } x \in [a + \delta, b - \delta], \\
\phi_{g_n}^{L} \left( \frac{x -(b-\delta)}{2\delta} \right) & \text{if } x \in [b - \delta, b + \delta], \\
0 & \text{if } x \notin [a - \delta, b + \delta].
\end{cases}
\]

(8)

Here, in order to make a PU function have a flat-top, we assume \( \delta \leq (b-a)/3 \). Let us note that \( \psi_{[a,b]}^{(\delta, n-1)}(x) \) is actually the convolution, \( \chi_{[a,b]}(x) * w_{\delta}^{n-1}(x) \), of the characteristic function \( \chi_{[a,b]} \) and the scaled window function \( w_{\delta}^{n-1} \), defined by (4) (Theorem 3.5 of [29]).

Since the two functions \( \phi_{g_n}^{R}, \phi_{g_n}^{L} \), defined by (6), satisfy the following relation:

\[
\phi_{g_n}^{R}(\xi) + \phi_{g_n}^{L}(\xi - 1) = 1, \text{ for } \xi \in [0, 1],
\]

(9)

if \( \varphi : [-\delta, \delta] \to [0, 1] \) is defined by

\[
\varphi(x) = (x + \delta)/(2\delta),
\]

then we have

\[
\phi_{g_n}^{R}(\varphi(x)) + \phi_{g_n}^{L}(\varphi(x) - 1) = 1, \text{ for } x \in [-\delta, \delta].
\]

5
Using the latter equation gives two basic one-dimensional $C^{n-1}$-functions

$$\psi_0^R(x) = \begin{cases} 
1 & \text{if } x \leq -\delta, \\
\phi_{g_n}^R(\frac{x-\delta}{2\delta}) & \text{if } x \in [-\delta, \delta], \\
0 & \text{if } x \geq \delta,
\end{cases} \quad (10)$$

$$\psi_0^L(x) = \begin{cases} 
0 & \text{if } x \leq -\delta \\
\phi_{g_n}^L(\frac{x+\delta}{2\delta}) & \text{if } x \in [-\delta, \delta], \\
1 & \text{if } x \geq \delta. 
\end{cases} \quad (11)$$

such that

$$0 \leq \psi_0^L(x), \psi_0^R(x) \leq 1, \quad \psi_0^R(x) + \psi_0^L(x) = 1, \text{ for all } x \in \mathbb{R}.$$ 

These PU functions (10) and (11) are extended to basic two-dimensional $C^{n-1}$-PU functions on $\mathbb{R}^2$ as follows:

Suppose $\overrightarrow{P_1P_2}$ is a straight line connecting two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with $x_1 \leq x_2$ such that $y_1 < y_2$ if $x_1 = x_2$. Then the angle between the positive $x$-axis and $\overrightarrow{P_1P_2}$ is determined by the following formula.

$$\theta = \begin{cases} 
\tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) & \text{if } x_2 \neq x_1, \\
\pi/2 & \text{if } x_2 = x_1.
\end{cases} \quad (12)$$

Let $T_{P_1P_2}$ be an affine transformation on $\mathbb{R}^2$ that transforms the straight line $\overrightarrow{P_1P_2}$ onto the $y$-axis defined by $T_{P_1P_2}(x, y) = (\tilde{x}, \tilde{y})$:

$$\begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix} = \begin{bmatrix} \cos(\pi/2 - \theta) & -\sin(\pi/2 - \theta) \\ \sin(\pi/2 - \theta) & \cos(\pi/2 - \theta) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}. \quad (13)$$
Figure 2: (a) Schematic diagram of basic PU functions, $\Psi^R_x = \Psi_{x=0}$ and $\Psi^L_x = 1 - \Psi_{x=0}$ in dimension two; (b) Transformed basic PU, $\Psi^R = \Psi_{P_1P_2}$ and $\Psi^L = \Psi_{P_1P_2}$, by the affine transformation $T_{P_1P_2}$.

Figure 3: By the lines $L_1, L_2, L_3, L_4$, the Domain $\Omega$ is partitioned into eight patches $Q_1, \cdots, Q_8$ and the flat-top parts of corresponding product PU functions are denoted by $Q_1^{flt}, \cdots, Q_8^{flt}$, respectively.
Then we define two PU functions by
\[ \Psi_{P_1P_2}(x, y) = \psi^R_0(\tilde{x}), \quad \Psi^*_P_{P_2}(x, y) = \psi^L_0(\tilde{x}) = 1 - \Psi_{P_1P_2}(x, y), \] (14)
that satisfy
\[ \Psi^*_P_{P_2}(x, y) + \Psi_{P_1P_2}(x, y) = 1, \quad \text{for all } (x, y) \in \mathbb{R}^2. \]
For example, if the line \( P_1P_2 \) is the \( y \)-axis (denoted by \( x = 0 \) in (15) and (16)), then the two-dimensional \( C_{\infty}^n \)-functions are
\[ \Psi_{x=0}(x, y) = \psi^R_0(x) \quad \text{and} \quad \Psi^*_P_{x=0}(x, y) = \psi^L_0(x), \quad \text{for all } (x, y) \in \mathbb{R}^2. \] (15)
In other words, two step-like-functions are the composition of the coordinate projection, \((x, y) \rightarrow x \), and \( \psi^R_0, \psi^L_0 \), respectively. The graph of \( \Psi_{x=0} \) (simply denoted by \( \Psi^R_0 \)) is sketched in Fig. 1. The schematic diagram for \( \Psi_{x=0} \) and \( \Psi^*_P_{x=0} \) is shown in Fig. 2. That is,
\[
\left\{ \begin{array}{ll}
\Psi^*_P_{x=0}(x, y) = 1 & \text{if } x \geq \delta \\
\Psi_{x=0}(x, y) = 0 & \text{if } x \leq -\delta \\
0 \leq \Psi^*_P_{x=0}(x, y) \leq 1 & \text{if } |x| \leq \delta
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll}
\Psi_{x=0}(x, y) = 0 & \text{if } x \geq \delta \\
\Psi^*_P_{x=0}(x, y) = 1 & \text{if } x \leq -\delta \\
0 \leq \Psi_{x=0}(x, y) \leq 1 & \text{if } |x| \leq \delta.
\right. \] (16)

2.3 Generalized product partition of unity

Suppose the given domain \( \Omega \) is partitioned into patches \( Q_j, j = 1, \cdots, n \) (background mesh) by lines and rays as shown in Fig. 3. Then the closed form partition of unity functions \( \Psi^P_j, j = 1, \cdots, n \) with flat-top, called \textit{generalized product partition of unity}, are introduced in [26]. In the following, we briefly review generalized product partition of unity for those patches shown in Fig. 3. Here \( \delta \) is a small number (usually in \([0.01, 0.1]\)) that depends on the sizes of patches. We describe the generalized product PU functions with a specific example in the following:

1. The triangular patch \( Q_1 \) of Fig. 3 is surrounded by lines \( L_1, L_2, L_4 \). Using (14), the step-like-basic PU functions of (15) on \( \mathbb{R}^2 \) are transformed onto lines \( L_1, L_2, L_4 \) to get three pairs of PU functions
\[
\Psi_{L_1}, \Psi^*_L_{L_1} := 1 - \Psi_{L_1} ; \quad \Psi_{L_2}, \Psi^*_L_{L_2} := 1 - \Psi_{L_2} ; \quad \Psi_{L_4}, \Psi^*_L_{L_4} := 1 - \Psi_{L_4}. \] \] (17)
2. The flat-top part of each patch \( Q_j \) that is outside the dotted lines is denoted by \( Q_{flt}^j \) on which only one of each pairs of basic PU functions is one.
3. Among the six functions in (17) related to lines enclosing \( Q_1 \), those which are one on \( Q_{flt}^1 \) are \( \Psi^*_L_{L_1}, \Psi_{L_2}, \) and \( \Psi^*_L_{L_4} \). A closed form PU function corresponding to the patch \( Q_1 \) is the product of these basic PU functions, that is, \( \Psi^P_1 = \Psi^*_L_{L_1} \cdot \Psi_{L_2} \cdot \Psi^*_L_{L_4} \).
4. Similarly, the closed form PU functions with wide flat-top corresponding to patches \( Q_j, j = 2, \cdots, 8 \), respectively, are
\[
\Psi^P_2 = \Psi^*_L_{L_1} \cdot \Psi_{L_2} \cdot \Psi^*_L_{L_4}, \quad \Psi^P_3 = \Psi_{L_1} \cdot \Psi^*_L_{L_2} ; \quad \Psi^P_4 = \Psi_{L_1} \cdot \Psi_{L_2} \cdot \Psi^*_L_{L_3};
\Psi^P_5 = \Psi_{L_1} \cdot \Psi_{L_3} \cdot \Psi^*_L_{L_4}, \quad \Psi^P_6 = \Psi_{L_1} \cdot \Psi_{L_4} ; \quad \Psi^P_7 = \Psi^*_L_{L_1} \cdot \Psi_{L_2} \cdot \Psi_{L_4},
\Psi^P_8 = \Psi^*_L_{L_2} \cdot \Psi_{L_4}. \]
Then, using the arguments similar to [26], one can show that
\[ \sum_{j=1}^{8} \Psi_j^{P}(x, y) = 1, \text{ for all } (x, y) \in \Omega. \]

These functions with flat-top are called the generalized product PU functions (we refer to [26] for the proof and the constructions for general cases).

It was shown in [26] that if a patch \( Q_j \) is a rectangle \([a, b] \times [c, d]\), then \( \Psi_j^{P} \) is the tensor product \( \psi^{(\delta,n-1)}_{[a,b]} \times \psi^{(\delta,n-1)}_{[c,d]} \), of one-dimensional functions defined by (8).

### 3 Reproducing polynomial particle methods

#### 3.1 Construction of RPP shape functions

Unlike existing meshfree particle methods such as the moving least squares methods and the reproducing kernel particle methods, we construct closed-form piecewise-polynomial particle shape functions with use of generalized product partition of unity.

(i) Suppose a domain \( \Omega \) is divided into quadrangular and triangular patches.

Let \( \hat{R} = [0, 1] \times [0, 1] \) (Fig. 6) be the reference rectangular patch and \( \hat{T} \) (Fig. 5) be the reference triangular patch. Let \( Q \) be a quadrangle with vertices \((a_i, b_i)\), \( i = 1, 2, 3, 4 \) and let \( T \) be a triangle with vertices \((a_i, b_i)\), \( i = 1, 2, 3 \). (Note that \( Q \) and \( T \) could be larger than physical patches as shown in Example 3 of Section 6.)

We define a bilinear mapping \( T(q) : \hat{R} \rightarrow Q \) by \( T(q) : \{(x, y)\} \)
\[
\begin{align*}
x &= a_1(1 - \xi)(1 - \eta) + a_2\xi(1 - \eta) + a_3\eta + a_4(1 - \xi)\eta \\
&= A_1 + A_2\xi + A_3\eta + A_4\xi\eta, \\
y &= b_1(1 - \xi)(1 - \eta) + b_2\xi(1 - \eta) + b_3\eta + b_4(1 - \xi)\eta \\
&= B_1 + B_2\xi + B_3\eta + B_4\xi\eta.
\end{align*}
\]

and a linear mapping \( T(t) : \hat{T} \rightarrow T \) by
\[
T(t) : \{(x, y)\} \begin{align*}
x &= a_1(1 - \xi - \eta) + a_2\xi + a_3\eta, \\
y &= b_1(1 - \xi - \eta) + b_2\xi + b_3\eta.
\end{align*}
\]

(ii) (a) Particles and particle shape functions on \( \hat{R} \): suppose \( \hat{\phi}_{ij}(\xi, \eta) = L_i(\xi) \times L_j(\eta), 0 \leq i, j \leq (k + 1) \), where \( L_i(\xi) \) is the Lagrange interpolating polynomial corresponding to the node \( \xi_i \) for \( i = 1, \cdots, (k + 1) \) and \( L_j(\eta) \) is also the Lagrange polynomial corresponding to the nodes \( \eta_j \), for \( j = 1, \cdots, (k + 1) \). Then we have
\[
\sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \xi_i^{\alpha_1}\eta_j^{\alpha_2}\hat{\phi}_{ij}(\xi, \eta) = \xi^{\alpha_1}\eta^{\alpha_2}, \text{ for integers } \alpha_1, \alpha_2 \text{ such that } 0 \leq \alpha_1, \alpha_2 \leq k.
\]

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(b) **Particles and particle shape functions on \( \hat{T} \):** Let \( n = (k + 1)(k + 2)/2 \). Then for \( s = 1, 2, \ldots, n \), particles \((\xi_s, \eta_s) \in \hat{T}\), and particle shape functions \( \phi_s \) are those shape functions for the conventional FEM. Then we have

\[
\sum_{s=1}^{n} \xi_s^{\alpha_1} \eta_s^{\alpha_2} \phi_s(\xi, \eta) = \xi^{\alpha_1} \eta^{\alpha_2}, \quad \text{for integers } \alpha_1, \alpha_2 \text{ such that } 0 \leq \alpha_1 + \alpha_2 \leq k. \tag{21}
\]

Then the RPP orders after these reference RPP shape functions are planted into the physical domain become as follows:

**Lemma 3.1.** Through the patch mappings, \( T_{(q)} \) or \( T_{(t)} \), particles as well as particle shape functions are planted in the physical domain \( \Omega \). Then there are some restrictions on the RPP order of the transformed particle shape functions on physical quadrangular patches:

1. Let \( (x, y) = T_{(q)}(\xi, \eta) \) and \( \phi_{ij} = \hat{\phi}_{ij} \circ T_{(q)}^{-1} \), for \( 0 \leq i, j \leq k \). Then

\[
\sum_{i=1}^{k+1} \sum_{j=1}^{k+1} x_i^{\alpha_1} y_j^{\alpha_2} \phi_{ij}(x, y) = x^{\alpha_1} y^{\alpha_2}, \quad \text{for } 0 \leq \alpha_1 + \alpha_2 \leq k. \tag{22}
\]

For example, even though \( \hat{\phi}_{ij}, 0 \leq i, j \leq k \), generate the monomial \( \xi^k \eta^k \), the transformed particle shape functions \( \phi_{ij}, 0 \leq i, j \leq k \), are not able to generate the monomial \( x^k y^k \).

2. Let \( (x, y) = T_{(t)}(\xi, \eta) \) and \( \phi_s = \hat{\phi}_s \circ T_{(t)}^{-1} \), for \( s = 1, 2, \ldots, n = (k + 1)(k + 2)/2 \). Then

\[
\sum_{s=1}^{n} x_s^{\alpha_1} y_s^{\alpha_2} \phi_s(x, y) = x^{\alpha_1} y^{\alpha_2}, \quad \text{for } 0 \leq \alpha_1 + \alpha_2 \leq k. \tag{23}
\]

**Proof.** (1)

\[
\sum_{i=1}^{k+1} \sum_{j=1}^{k+1} x_i^{\alpha_1} y_j^{\alpha_2} \phi_{ij}(x, y) = \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} [A_1 + A_2 \xi_i + A_3 \eta_j + A_4 \xi_i \eta_j]^{\alpha_1} [B_1 + B_2 \xi_i + B_3 \eta_j + B_4 \xi_i \eta_j]^{\alpha_2} \hat{\phi}_{ij}(\xi, \eta)
\]

If \( 0 \leq \alpha_1 + \alpha_2 \leq k \), by the relation (20), each term of \( [A_1 + A_2 \xi_i + A_3 \eta_j + A_4 \xi_i \eta_j]^{\alpha_1} [B_1 + B_2 \xi_i + B_3 \eta_j + B_4 \xi_i \eta_j]^{\alpha_2} \) is generated by \( \hat{\phi}_{ij}(\xi, \eta), 0 \leq i, j \leq k \). Hence the last equation equals to

\[
[A_1 + A_2 \xi + A_3 \eta + A_4 \xi \eta]^{\alpha_1} [B_1 + B_2 \xi + B_3 \eta + B_4 \xi \eta]^{\alpha_2} = x^{\alpha_1} y^{\alpha_2}.
\]

(2) Since \( T_{(t)} \) is a linear mapping, \( T_{(t)}^{-1} \) can be constructed explicitly and hence the relation (23) is obvious. □
(iii) Let $\omega^l_\delta$ be the support of the generalized product PU function $\Psi_P^l$ corresponding to the $l$-th patch in the domain $\Omega$ for each $l = 1, \cdots, N$. Let $\phi^{(l)}_{ij}$ be particle shape functions corresponding to the particles that fall into $\omega^l_\delta$ by the patch mapping.

Then, for each $l$, $l = 1, \cdots, N$, the approximation functions $\Psi_P^l \cdot \phi^{(l)}_{ij}, 1 \leq i, j \leq (k + 1)$ (when the $l$-th patch is rectangular) and $\Psi_P^l \cdot \phi^{(l)}_s, 1 \leq s \leq (k + 1)(k + 2)/2$ (when the $l$-th patch is triangular) are smooth closed-form piecewise-polynomials and have the compact support $\omega^l_\delta$.

Reproducing Polynomial Particle Methods (RPPM) proposed in this paper is a meshfree particle method and it is actually the Galerkin approximation methods using the local approximation functions constructed above for global basis functions. Let us note that these basis functions satisfy the Kronecker delta property and have the polynomial reproducing property of order $k$.

3.2 Error estimates

Since the local approximation functions corresponding the particles planted through the patch mappings have the reproducing polynomial property whenever no essential boundary conditions are imposed, the proposed method is a meshfree particle method. Whereas, since the method employs the generalized product partition of unity for the construction of approximation functions, it is also a PUFEM. Thus, we can estimate the errors of approximate solutions in two different approaches: an estimate for meshfree methods similar to [13] and an estimate for PUFEM similar to [25].

In this section, modifying the error estimates for PUFEM due to [3] and [25], we have the following extended error estimates for our method.

**Theorem 3.1.** Let $\Omega$ be a two-dimensional polygonal domain. Let $\Psi_P^l$ be a generalized product partition of unity corresponding to a patch $Q_l$, for $l = 1, \cdots, N$, and $\omega_l$ be the support of $\Psi_P^l$ such that

$$0 \leq \Psi_P^l \leq 1 \text{ on } \omega_l^{\text{non-flat}}; \quad \Psi_P^l = 1 \text{ on } \omega_l^{\text{flat}};$$

$$\omega_l = \omega_l^{\text{flat}} \cup \omega_l^{\text{non-flat}} \quad \text{(note: } \omega_l^{\text{flat}} \text{ is also denoted by } Q_l^{\text{flat}} \text{ in Figs. } 3, 8, 9) \text{.}$$

We assume that $\text{card}\{i|x \in \omega_i\} \leq M$ for all $x \in \Omega$. Let a collection of local approximation spaces $\mathcal{V}_l \subset H^2(\Omega \cap \omega_l)$ be given. Let $u \in H^2(\Omega)$ be the function to be approximated. Assume that the local approximation spaces $\mathcal{V}_l$ have the following approximation properties: on each patch $\Omega \cap \omega_l$, the function $u$ can be approximated by a function $v_l \in \mathcal{V}_l$ such that

$$|u - v_l|_{m, \omega_l} \leq \varepsilon_t^{(m)}, \quad \text{and} \quad |u - v_l|_{m, \infty, \omega_l^{\text{non-flat}}} \leq \varepsilon_t^{(m)}, \quad \text{for } m = 0, 1, 2,$$

hold for all $l$. Let $v_l = \sum_{k=1}^{N_l} c_k \phi_k^{(l)} \in \mathcal{V}_l$, ($\phi_k^{(l)}$ is either $\phi_{ij}^{(l)}$ in (22) or $\phi_s^{(l)}$ in (23)). Then the function

$$u_{\text{app}} := \sum_{l=1}^{N} (\Psi_P^l \cdot v_l) \in \mathcal{V}_{\text{RPP}},$$
satisfies the global estimates

\begin{align*}
(i) & \left| |u - u^{\text{app}}| \right|_{0;\Omega}^2 \leq M \sum_{l=1}^{N} (\varepsilon_l^{(0)})^2, \\
(ii) & \left| |(u - u^{\text{app}})| \right|_{1;\Omega}^2 \leq 2M \sum_{l=1}^{N} \left\{ (\varepsilon_l^{(0)})^2 (|\Psi_l^P|_{1,\omega_{l}^{n,\text{flat}}})^2 + (\varepsilon_l^{(1)})^2 \right\}, \\
(iii) & \left| |(u - u^{\text{app}})| \right|_{2;\Omega}^2 \leq 4M \sum_{l=1}^{N} \left\{ (\varepsilon_l^{(0)})^2 (|\Psi_l^P|_{2,\omega_{l}^{n,\text{flat}}})^2 + (\varepsilon_l^{(2)})^2 + 5(|\Psi_l^P|_{1,\omega_{l}^{n,\text{flat}}})^2 \right\}.
\end{align*}

Proof. The proof proceeds by using similar arguments as those of Theorem 6.2 of [3].

For brevity, we use the following notations: \( \partial_x u = u_x \), \( (u - v_l) = e_l \), and \( \Psi_l^P = \Psi_l \).

Using the fact that the derivatives of \( \Psi_l \) are zero except for a small \( 2\delta \)-width strip around the support of \( \Psi_l \) and the inequalities, \( (a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2) \), we have the following error estimates:

(ii) The approximation error in the semi 1-norm is bounded as follows:

\[
\left| |(u - u^{\text{app}})| \right|_{1;\Omega}^2 = \left| \sum_{l=1}^{N} \Psi_l(u - v_l) \right|_{1;\Omega}^2 = \int_{\Omega} \left\{ \left[ (\sum_{l} \Psi_l e_l)_{x} \right]^2 + \left[ (\sum_{l} \Psi_l e_l)_{y} \right]^2 \right\} \\
\leq 2 \int_{\Omega} \left\{ \left[ \sum_{l} (\Psi_{l,x} e_l) \right]^2 + \left[ \sum_{l} (\Psi_{l,y} e_l) \right]^2 \right\} \\
\leq 2M \int_{\Omega} \left\{ \left[ (\Psi_{l,x} e_l)^2 + (\Psi_{l,y} e_l)^2 \right] + \left[ (\Psi_{l,x} e_l)^2 + (\Psi_{l,y} e_l)^2 \right] \right\} \\
\leq 2M \int_{\Omega} \left\{ \left[ (\Psi_{l,x} e_l)^2 + (\Psi_{l,y} e_l)^2 \right] (e_l)^2 \right\} + \left[ (\Psi_{l,x})^2 (e_l)^2 + (\Psi_{l,y})^2 (e_l)^2 \right] \\
\leq 2M \sum_{l} \left\{ (|e_l|_{0,\omega_{l}^{n,\text{flat}}})^2 (|\Psi_l|_{1,\omega_{l}^{n,\text{flat}}})^2 + (|e_l|_{1,\omega_{l}^{n,\text{flat}}})^2 \right\} \quad \text{(since } 0 \leq \Psi_l \leq 1 \text{ on } \Omega). 
\]

(iii) The approximation error in the semi 2-norm is bounded as follows:
\[(u - u^{app})^2_{2,\Omega} = |\sum_{l=1}^N \Psi_l(u - v_l)|^2_{2, \Omega} = \int_{\Omega} \left\{ \left[ \sum_l \Psi_{l,xx} e_l \right]^2 + \left[ \sum_l \Psi_{l,xy} e_l \right]^2 + \left[ \sum_l \Psi_{l,yy} e_l \right]^2 \right\} \]

\[
\leq 3 \int_{\Omega} \left\{ \left[ \sum_l \Psi_{l,xx} e_l \right]^2 + \left[ 2 \sum_l \Psi_{l,x} e_{l,x} \right]^2 + \left[ \sum_l \Psi_{l,xx} e_l \right]^2 \right\} + 4 \int_{\Omega} \left\{ \left[ \sum_l \Psi_{l,xy} e_l \right]^2 + \left[ \sum_l \Psi_{l,y} e_{l,y} \right]^2 + \left[ \sum_l \Psi_{l,xx} e_l \right]^2 \right\} + 3 \int_{\Omega} \left\{ \left[ \sum_l \Psi_{l,yy} e_l \right]^2 + \left[ 2 \sum_l \Psi_{l,y} e_{l,y} \right]^2 + \left[ \sum_l \Psi_{l,yy} e_l \right]^2 \right\} 
\]

\[
\leq 3M \int_{\Omega} \sum_l \left\{ |\Psi_{l,xx} e_l|^2 + |2\Psi_{l,x} e_{l,x}|^2 + |\Psi_{l,xx} e_l|^2 \right\} + 4M \int_{\Omega} \sum_l \left\{ |\Psi_{l,xy} e_l|^2 + |\Psi_{l,y} e_{l,y}|^2 + |\Psi_{l,xy} e_l|^2 \right\} + 3M \int_{\Omega} \sum_l \left\{ |\Psi_{l,yy} e_l|^2 + |2\Psi_{l,y} e_{l,y}|^2 + |\Psi_{l,yy} e_l|^2 \right\} 
\]

\[
\leq 4M \int_{\Omega} \sum_l \left\{ e_l^2 (\Psi_{l,xx}^2 + \Psi_{l,yy}^2 + \Psi_{l,xy}^2) + \Psi_l^2 (e_{l,xx} + e_{l,yy})^2 + (\Psi_{l,xy} e_{l,y})^2 + (\Psi_{l,y} e_{l,x})^2 \right\} 
\]

\[
\leq 4M \sum_l \left\{ \left| e_l \right|_{0,\infty,\omega_i^{n-flat}} (\left| \Psi_l \right|_{2,\omega_i^{n-flat}})^2 + \left( e_l \right|_{2,\omega_i}^2 + 5 (\left| \Psi_l \right|_{1,\omega_i^{n-flat}})^2 \right\} 
\]

(\text{using } 0 \leq \Psi_l \leq 1 \text{ at the last step}).

The proof of part (i) is similar to that of part (ii).

\[ \Box \]

For a small \( \delta \), the absolute value of the derivatives of \( \Psi_l^P \) become large (\( O(1/\delta) \)), however, since they are non-zero only along the 2\( \delta \)-width strip along the boundary of its support \( \omega_i \), \( |\Psi_l^P|_{m,\omega_i^{n-\text{flat}},m = 1,2} \) are not too large as shown in Table 1. For example, in order to get \( C^1 \)-generalized product PU functions, suppose \( \phi_{g_2}^R(\frac{x + \delta}{2\delta}) = \left[ 1 - \frac{x + \delta}{2\delta} \right]^2 \left[ 1 + 2(\frac{x + \delta}{2\delta}) \right] \) is used in the definition (10). Then

\[
\int_{-\delta}^{\delta} \left[ \partial_x \phi_{g_2}^R(\frac{x + \delta}{2\delta}) \right]^2 dx = \frac{3}{\delta^3}, \quad \int_{-\delta}^{\delta} \left[ \partial_{xx} \phi_{g_2}^R(\frac{x + \delta}{2\delta}) \right]^2 dx = \frac{3}{2\delta^5}. 
\]

Hence, for this choice of PU functions, we have

\[
|\Psi_l^P|_{1,\omega_i^{n-\text{flat}}} \leq \left( L \frac{3}{\delta^3} \right)^{1/2}, \quad |\Psi_l^P|_{2,\omega_i^{n-\text{flat}}} \leq \left( L \frac{3}{2\delta^5} \right)^{1/2}, \quad (28)
\]

where \( L \) is the length of the perimeter of the patch \( Q_l \) corresponding to \( \Psi_l^P \). The upper bounds are in Table 1 for various \( \delta \).
Table 1: Bound for $|\Psi^P_t|_{1,\omega_i^{n-flat}} \leq [3L/(5\delta)]^{1/2}$ and $|\Psi^P_t|_{2,\omega_i^{n-flat}}[3L/(2\delta^3)]^{1/2}$ for various $\delta$, where $L$ is the length of the perimeter of patch $Q_i$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>0.1</th>
<th>0.05</th>
<th>0.025</th>
<th>0.01</th>
<th>0.0075</th>
<th>0.005</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[3/(5\delta)]^{1/2}$</td>
<td>2.45</td>
<td>3.46</td>
<td>4.90</td>
<td>7.75</td>
<td>8.94</td>
<td>10.95</td>
</tr>
<tr>
<td>$[3/(2\delta^3)]^{1/2}$</td>
<td>38.73</td>
<td>109.54</td>
<td>309.84</td>
<td>1224.74</td>
<td>1885.62</td>
<td>3464.10</td>
</tr>
</tbody>
</table>

Figure 4: The 3-dimensional plate $\Omega$ and 2-dimensional midplane $\Omega$
4 Models for elastic plates

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary, which represents the mid-plane of a plate, which we assume to be of thickness $d$ ($d \ll \text{diam}(\Omega)$) (see, Fig. 4). We represent the 3-dimensional plate as $\hat{\Omega} = \{(x, y, z) \in \mathbb{R}^3 | (x, y) \in \Omega, |z| < d/2\}$.

Under some hypothesis, the 3-dimensional elasticity equations for plates are reduced to 2-dimensional equations (plate models). The popular plate models are the Kirchhoff plate model for thin plates, and for moderately thick plates, first order shear deformation plate model (known as the Reissner-Mindlin plate model), third order shear deformation plate model, and higher order plate models for higher accuracy.

4.1 Classical plate theory

The classical (Kirchhoff) plate theory is one in which the displacement field is based on the Kirchhoff hypothesis: (1) Straight lines perpendicular to the middle surface before deformation remain straight after deformation; (2) The transverse normals do not experience elongation (i.e. they are inextensible); (3) The transverse normals remain perpendicular to the middle surface after deformation ([33],[36],[37]). From this hypothesis, $\epsilon_{zz} = 0$ and the transverse shear strains are zero: $\epsilon_{xz} = \epsilon_{yz} = 0$. Suppose $(u, v, w)$ denote the total displacement of a point along the $xyz$-coordinate system and $(u_0, v_0, w_0)$ denote the values of $u, v$ and $w$ at the point $(x, y, 0)$. Then these conditions imply

$$u(x, y, z) = u_0 - z \frac{\partial w}{\partial x}, \quad v(x, y, z) = v_0 - z \frac{\partial w}{\partial y}, \quad w(x, y, z) = w_0. \tag{29}$$

Substituting the displacement functions of (29) into the virtual work formulation, we have ([36],[37])

$$B(w, v) = \mathcal{F}(v), \text{ for } w, v \in H^2(\Omega), \tag{30}$$

where

$$B(w, v) = D \int_\Omega \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2} \end{bmatrix}^T \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1 - \nu) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 v}{\partial x^2} \\ \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial y^2} \end{bmatrix} dxdy, \tag{31}$$

$$\mathcal{F}(v) = \int_\Omega p(x, y)vdxdy + \int_\Gamma M_n \frac{\partial v}{\partial n} dt - \int_\Gamma \left( Q_n + \frac{\partial M_{nt}}{\partial t} \right) v dt. \tag{32}$$
Table 2: Boundary conditions in the classical theory of plate

<table>
<thead>
<tr>
<th>Type of support</th>
<th>Essential (Geometric) B.C.</th>
<th>Natural (Force) B.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>clamped</td>
<td>$w = 0$, $\partial w/\partial n = 0$</td>
<td>None</td>
</tr>
<tr>
<td>simple support</td>
<td>$w = 0$</td>
<td>$M_n = 0$</td>
</tr>
<tr>
<td>free</td>
<td>None</td>
<td>$M_n = M_{nt} = Q_n = 0$</td>
</tr>
<tr>
<td>symmetry</td>
<td>$\partial w/\partial n = 0$</td>
<td>$Q_n + \partial M_{nt}/\partial t = 0$</td>
</tr>
<tr>
<td>antisymmetry</td>
<td>$w = 0$</td>
<td>$M_n = 0$</td>
</tr>
</tbody>
</table>

where $\nu$ is the Poisson’s ratio and, $E$ is the Young’s modulus of an isotropic elastic material, and

$$D = \frac{Ed^3}{12(1-\nu^2)},$$

$$M_x = D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \quad M_y = D \left( \nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right), \quad M_{xy} = D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} = -M_{yx},$$

$$Q_x = D \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right), \quad Q_y = D \left( \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \right), \quad Q_n = Q_x \cos \alpha + Q_y \sin \alpha,$$

$$M_n = [M_x, M_y, M_{xy} - M_{yx}][\cos^2 \alpha, \sin^2 \alpha, \sin \alpha \cos \alpha]^T,$$

$$M_{nt} = [-M_x + M_y, M_{xy}][\sin \alpha \cos \alpha, (\cos^2 \alpha - \sin^2 \alpha)]^T,$$

$$p = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}.$$

Here $\alpha$ is the angle between the $x$-axis and the normal axis of the normal-tangential coordinate system as shown in Fig. 4. The conventional boundary conditions in the classical theory of plate are listed in Table 2.

Let $\mathcal{V}^{RPP}$ be an approximation space constructed by use of the generalized partition of unity functions $\Psi^P_l$ and the reference shape functions $\hat{\phi}_k$ constructed in the previous sections. That is,

$$\mathcal{V}^{RPP} = \text{span}\{\Psi^P_l \cdot [\hat{\phi}_k \circ T^{-1}_l] : l = 1, 2, \cdots , N; k = 1, 2, \cdots , N_l\},$$

(33)

where $\Omega$ is partitioned into the $N$ numbers of patches $Q_1, \cdots , Q_N$ and $T_l$ is the patch mapping from a reference patch into $\omega^P_l$, the support of the generalized product PU function $\Psi^P_l$, which is associated with the physical patch $Q_l$. Then the approximation space $\mathcal{V}^{RPP}$ has the following properties: (1) high regularity of each member; (2) the Kronecker delta property at almost all particles; (3) reproducing polynomial property of high order on the patches with no intersections with clamped boundaries.

Now, the proposed meshfree particle method (RPPM) for plate problems is the Galerkin method with use of $\mathcal{V}^{RPP}$ as follows: Find $w \in \mathcal{V}^{RPP}$ such that

$$\mathcal{B}(w, v) = \mathcal{F}(v), \text{ for all } v \in \mathcal{V}^{RPP}.$$
Figure 5: Reference patches and the degree of freedom linear forms on $\mathcal{P}_3(\hat{T})$ and $\mathcal{P}_4(\hat{T})$ whose dual basis functions clamp one edge. "•" denotes an evaluation at the point and "<---" indicates an evaluation of the $\xi$-derivative at the point.

Figure 6: The degree of freedom linear forms on $\mathcal{Q}_3(\hat{R})$ and $\mathcal{Q}_4(\hat{R})$ whose dual basis functions clamp one edge. "•" denotes an evaluation at the point and "<---" indicates an evaluation of the $\xi$-derivative at the point.
5 Particle shape functions with Kronecker delta property to deal with essential boundary conditions of plates

In this section, \( \hat{T} \) and \( \hat{R} \), respectively, denote the reference triangle with vertices (0, 0), (1, 0), (0, 1), and the reference rectangle \([0, 1] \times [0, 1]\) shown in Figs. 5 and 6. \( \mathcal{P}_n(\hat{T}) = \text{span}\{\xi^i \eta^j | 0 \leq i, j \leq n, 0 \leq i + j \leq n\}\) is the space of all polynomials of total degree less than or equal to \( n \). \( \mathcal{Q}_m(\hat{R}) = \text{span}\{\xi^i \eta^j | 0 \leq i, j \leq m\}\) is the space of all polynomials of degree less than or equal to \( m \) in each variable. Hence, \( \dim(\mathcal{P}_n(\hat{T})) = \binom{n+1}{2} \) and \( \dim(\mathcal{Q}_m(\hat{R})) = \binom{m+1}{2} \). \( V^* \) denotes the dual vector space of a vector space \( V \).

5.1 Shape functions for imposing simple support boundary condition

Without loss of generality, we consider the construction of patchwise smooth RPP shape functions with compact support that have the polynomial reproducing order 3.

[RPP functions on the reference Triangular patch \( \hat{T} \)]

The nodal shape functions on the reference triangle \( \hat{T} \) corresponding to vertices, (0, 0), (1, 0), (0, 1), respectively, are \( L_1(\xi, \eta) = 1 - \xi - \eta \), \( L_2(\xi, \eta) = \xi \), \( L_3(\xi, \eta) = \eta \). The following ten functions are the Lagrange interpolating shape functions of RPP order 3 that correspond to three vertices, six lateral nodes, and one interior node, respectively:

\[
\begin{align*}
\hat{\phi}_1 &= L_1(3L_1 - 1)(3L_1 - 2)/2; \\
\hat{\phi}_2 &= L_2(3L_2 - 1)(3L_2 - 2)/2; \\
\hat{\phi}_3 &= L_3(3L_3 - 1)(3L_3 - 2)/2; \\
\hat{\phi}_4 &= (9/2)L_1L_2(3L_1 - 1); \\
\hat{\phi}_5 &= (9/2)L_1L_3(3L_1 - 1); \\
\hat{\phi}_6 &= (9/2)L_2L_1(3L_2 - 1); \\
\hat{\phi}_7 &= (9/2)L_2L_3(3L_2 - 1); \\
\hat{\phi}_8 &= (9/2)L_3L_1(3L_3 - 1); \\
\hat{\phi}_9 &= (9/2)L_3L_2(3L_3 - 1); \\
\hat{\phi}_{10} &= 27L_1L_2L_3.
\end{align*}
\] (35)

In what follows, the \( k \)-th Lagrange interpolating polynomial of degree \( n - 1 \) associated with \( n \)-distinct nodes \( \xi_1, \cdots, \xi_n \) is denoted by \( L_{n,k}(\xi) = \prod_{i=1,i \neq k}^n \frac{\xi - \xi_i}{\xi_k - \xi_i} \).

[RPP functions on the reference rectangular patch \( \hat{R} \) shown in Fig. 6]
Let $L_{4,j}, j = 1, 2, 3, 4,$ be the Lagrange interpolating polynomials of order 3 corresponding to nodes $0, 1/3, 2/3, 1,$ respectively. Then, we have 16 RPP shape functions of order 3:

$$
\hat{\phi}_k(\xi, \eta) = L_{4,i}(\xi) \cdot L_{4,j}(\eta),
$$

$k = 4(i - 1) + j, 1 \leq i, j \leq 4,$ \hfill (36)

[Local approximation functions defined on the domain $\Omega$]

Those RPP shape functions defined by (35) and (36) can be planted on a physical domain $\Omega$ by proper patch mappings to make smooth local approximation functions on triangular or quadrangular patches in $\Omega$. For example, let $T_l : \hat{R} \rightarrow Q_l$ be a smooth patch mapping and $\Psi_l^P(x,y)$ be the generalized product $PU$ corresponding to $Q_l$. Then

$$
\Psi_l^P(x,y) \cdot \hat{\phi}_k(T_l^{-1}(x,y)), \quad k = 1, \cdots, 16
$$

are smooth functions with compact support $\omega_l = \text{supp}\Psi_l^P \supset Q_l$.

5.2 Shape functions for implementing clamped boundary conditions

The proofs of three technical Lemmas presented in this section can be found in appendix.

5.2.1 Triangular reference patches

**Lemma 5.1.** Suppose the linear functionals (called the degree of freedom linear forms) $N_i$ on $P_3(\hat{T})$ are defined by

$$
\begin{align*}
N_1(f) &= f(0,0), & N_2(f) &= f(0,1/3), & N_3(f) &= f(0,2/3), & N_4(f) &= f(0,1), \\
N_5(f) &= f(1/2,0), & N_6(f) &= f(1/2,1/2), & N_7(f) &= f(1,0), \\
N_8(f) &= \frac{\partial f}{\partial \xi}(0,0), & N_9(f) &= \frac{\partial f}{\partial \eta}(0,1/3), & N_{10}(f) &= \frac{\partial f}{\partial \eta}(0,1),
\end{align*}
$$

for $f \in P_3(\hat{T})$. Then we have the following:

(i) \{ $N_i : i = 1, \cdots, 10$ \} is a basis of the dual space $[P_3(\hat{T})]^*$. \\
(ii) If a polynomial $f \in P_3(\hat{T})$ satisfies $N_j(f) = 0, \text{ for } j = 1, 2, 3, 4, 8, 9, 10,$ then $f \cdot \frac{\partial f}{\partial \xi}$ and $\frac{\partial f}{\partial \eta}$ are identically zero along the line $\xi = 0$.

Let \{ $\hat{\phi}_j, j = 1, \cdots, 10$ \} be the dual basis on $P_3(\hat{T})$ of the basis \{ $N_j, j = 1, \cdots, 10$ \} such that for each $i = 1, \cdots, 10, \quad N_i(\hat{\phi}_j) = \delta_i^j$ for $j = 1, \cdots, 10$. Then it follows from Lemma 5.1 that for $k = 5, 6, 7,$

$$
\hat{\phi}_k = 0, \quad \frac{\partial \hat{\phi}_k}{\partial \xi} = 0, \quad \frac{\partial \hat{\phi}_k}{\partial \eta} = 0 \hfill (38)
$$

along the $\eta$-axis.

Suppose $T$ is a patch mapping from the reference patch $\hat{T}$ onto a physical patch $\Omega_T$. Then we have

$$
\nabla_{xy}\hat{\phi}_k(T^{-1}(x,y)) = \nabla_{\xi\eta}\hat{\phi}_k(T^{-1}(x,y)) \cdot J(T^{-1})(x,y) \hfill (39)
$$

Since $\nabla_{\xi\eta}\hat{\phi}_k(0, \eta) = 0$ for $k = 5, 6, 7, \quad \phi_k \equiv \hat{\phi}_k \circ T^{-1}$ and $\frac{\partial \phi_k}{\partial x}, \frac{\partial \phi_k}{\partial y}$, are zero along the side of $\Omega_T$ connecting the vertices $T(0,0)$ and $T(0,1)$. 

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5.2.2 Extension to the degree of freedom linear functionals on $\mathcal{P}_n(\hat{T})$ for $n \geq 4$

Lemma 5.1 can be extended to the construction of the degree of freedom linear functionals for any order $n$.

For example, suppose the 15 degree of freedom linear functionals on $\mathcal{P}_4(\hat{T})$ are as shown in Fig. 5. That is, the linear functionals are defined as follows: for $f = \sum_{0 \leq i+j \leq 4} a_{ij} \xi^i \eta^j \in \mathcal{P}_4(\hat{T})$,

$$
\begin{align*}
N_1(f) &= f(0,0), & N_2(f) &= f(0,1/4), & N_3(f) &= f(0,2/4), & N_4(f) &= f(0,3/4), \\
N_5(f) &= f(0,1), & N_6(f) &= f(1/3,0), & N_7(f) &= f(1/3,1/3), & N_8(f) &= f(1/3,2/3), \\
N_9(f) &= f(2/3,0), & N_{10}(f) &= f(2/3,1/3), & N_{11}(f) &= f(1,0), \\
N_{12}(f) &= \frac{\partial f}{\partial \xi}(0,0), & N_{13}(f) &= \frac{\partial f}{\partial \xi}(0,1/4), & N_{14}(f) &= \frac{\partial f}{\partial \xi}(0,2/4), & N_{15}(f) &= \frac{\partial f}{\partial \xi}(0,1).
\end{align*}
$$

(40)

By using a similar argument to Lemma 5.1, one can show that if $N_j(f) = 0$, for all $j = 1, \cdots, 15$, then $f \equiv 0$ on $\mathbb{R}^2$. Hence $N_j$ defined by (40) are a dual basis of $\mathcal{P}_4(\hat{T})$.

5.2.3 Reference rectangular patch

**Lemma 5.2.** (Shape functions clamping one side)

Suppose the linear forms (the degree of freedom linear functionals) $N_i$ on $\mathcal{Q}_3(\hat{R}) = \text{span}\{\xi^i \eta^j | 0 \leq i, j \leq 3\}$ are defined by

$$
\begin{align*}
N_1(f) &= f(0,0), & N_2(f) &= f(0,1/3), & N_3(f) &= f(0,2/3), & N_4(f) &= f(0,1), \\
N_5(f) &= f(1/2,0), & N_6(f) &= f(1/2,1/3), & N_7(f) &= f(1/2,2/3), & N_8(f) &= f(1/2,1), \\
N_9(f) &= f(1,0), & N_{10}(f) &= f(1,1/3), & N_{11}(f) &= f(1,2/3), & N_{12}(f) &= f(1,1), \\
N_{13}(f) &= \frac{\partial f}{\partial \xi}(0,0), & N_{14}(f) &= \frac{\partial f}{\partial \xi}(0,1/3), & N_{15}(f) &= \frac{\partial f}{\partial \xi}(0,2/3), & N_{16}(f) &= \frac{\partial f}{\partial \xi}(0,1),
\end{align*}
$$

(41)

for $f \in \mathcal{Q}_3(\hat{R})$. Then the 16 linear functionals have the following properties:

(i) $\{N_i : i = 1, \cdots, 16\}$ is a basis of the dual space $[\mathcal{Q}_3(\hat{R})]^*$.

(ii) Suppose $f \in \mathcal{Q}_3(\hat{R})$ satisfies $N_j(f) = 0$ for $j = 1, 2, 3, 4, 13, 14, 15, 16$, then $f, \frac{\partial f}{\partial \eta}$ and $\frac{\partial f}{\partial \eta}$ are identically zero along the line $\xi = 0$.

**Lemma 5.3.** (Shape functions clamping two sides)

Suppose the linear forms (the degree of freedom linear functionals) $N_i$ on $\mathcal{Q}_3(\hat{R}) = \text{span}\{\xi^i \eta^j | 0 \leq i, j \leq 3\}$ are defined by

$$
\begin{align*}
N_1(f) &= f(0,0), & N_2(f) &= f(0,1/3), & N_3(f) &= f(0,2/3), & N_4(f) &= f(0,1), \\
N_5(f) &= f(1/2,0), & N_6(f) &= f(1/2,1/3), & N_7(f) &= \frac{\partial f}{\partial \eta}(1/2,0), & N_8(f) &= f(1/2,1), \\
N_9(f) &= f(1,0), & N_{10}(f) &= f(1,1/3), & N_{11}(f) &= \frac{\partial f}{\partial \eta}(1,0), & N_{12}(f) &= f(1,1), \\
N_{13}(f) &= \frac{\partial f}{\partial \xi}(0,0), & N_{14}(f) &= \frac{\partial f}{\partial \xi}(0,1/3), & N_{15}(f) &= \frac{\partial f}{\partial \xi}(0,2/3), & N_{16}(f) &= \frac{\partial f}{\partial \xi}(0,1),
\end{align*}
$$

(42)

for $f \in \mathcal{Q}_3(\hat{R})$. Then the 16 linear functionals have the following properties:

(i) $\{N_i : i = 1, \cdots, 16\}$ is a basis of the dual space $[\mathcal{Q}_3(\hat{R})]^*$.

(ii) Suppose $f \in \mathcal{Q}_3(\hat{R})$ satisfies $N_j(f) = 0$ for $j = 1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 15, 16$, then $f = \frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0$ along the the lines $\xi = 0, \eta = 0$. 

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5.2.4 Expression of dual basis of order 3 for the clamped boundary conditions

Let the dual basis of $N_k$ be denoted by $\hat{\phi}_k$. Then, by using the uniqueness of the dual basis $\hat{\phi}_k$ with $N_k(\hat{\phi}_j) = \delta_{ij}$ we are able to specifically determine the basis functions $\hat{\phi}_k$ that have the Kronecker delta property. However, these functions designed to impose clamped boundary conditions no longer have the polynomial reproducing property.

[A1: One-edge-clamped particle shape functions of order 3 (Fig. 6)] Let $\{\hat{\phi}_j, j = 1, \cdots, 16\}$ be the dual basis on $Q_3(\bar{R})$ of the basis $\{N_j, j = 1, \cdots, 16\}$ such that for each $i = 1, \cdots, 16$, $N_i(\hat{\phi}_j) = \delta_{ij}$, for $j = 1, \cdots, 16$. Then it follows from Lemma 5.2 that for $k = 5, 6, 7, 8, 9, 10, 11, 12$, $\phi_k = 0$, $\partial \hat{\phi}_k / \partial \xi = 0$, $\partial \hat{\phi}_k / \partial \eta = 0$ along the line $\xi = 0$.

Now, $\hat{\phi}_k, k = 5, 6, 7, 8, 9, 10, 11, 12$ can be written as follows: Let $L_{3,2}(\xi), L_{3,3}(\xi)$ be the second and the third Lagrange interpolating polynomials associated with three nodes 0, 1/2, 1. For $k = 1, 2, 3, 4$, let $L_{4,k}(\eta)$ be the $k$-th Lagrange interpolating polynomial associated with nodes 0, 1/3, 2/3, 1. Since $L_{3,3}(0) = L_{3,3}(0) = 0$, $\xi L_{3,2}(\xi) = -4\xi^2(\xi - 1)$, and $\xi L_{3,3}(\xi) = -2\xi^2(\xi - 1/2)$. Therefore, $\xi L_{3,2}(\xi), \xi L_{3,3}(\xi)$ and their $\xi$-derivatives are vanishing along the $\eta$-axis. Thus, we have

$$\left\{ \begin{array}{ll}
\text{For } k = 5, 6, 7, 8, & \hat{\phi}_k(\xi, \eta) = c_k \xi L_{3,3}(\xi) \cdot L_{4,k-4}(\eta), \\
\text{For } k = 9, 10, 11, 12, & \hat{\phi}_k(\xi, \eta) = c_k \xi L_{3,3}(\xi) \cdot L_{4,k-8}(\eta),
\end{array} \right. \quad (43)$$

where $c_k$ is the normalizing constant that makes $N_k(\hat{\phi}_k) = 1$.

[A2: Two-edges-clamped particle shape functions of order 3 (Fig. 7)] $\hat{\phi}_k, k = 6, 8, 10, 12$ can be written specifically as follows: For $k = 2, 3$, let $L_{3,k}(\eta)$ be the $k$-th Lagrange interpolating polynomials of degree 2 associated with nodes 0, 1/3, 1. Then $L_{3,2}(\eta) = -9(2/3)\eta(\eta - 1)$ and $L_{3,3}(\eta) = (3/2)\eta(\eta - 1/3)$. Hence, $\eta L_{3,2}(\eta), \eta L_{3,3}(\eta)$ and their $\eta$-derivatives are vanishing along the $\xi$-axis. Thus, we have

$$\left\{ \begin{array}{ll}
\text{For } k = 10, 12, & \hat{\phi}_k(\xi, \eta) = c_k \xi L_{3,2}(\xi) \cdot \eta L_{3,3}(\eta), \text{ for } j = 2, 3, \text{ respectively;}
\end{array} \right. \quad (44)$$

where $c_k$ is the normalizing constant that makes $N_k(\hat{\phi}_k) = 1$.

5.2.5 Expression of the dual basis of higher order for the clamped boundary conditions

Lemma 5.2 (one-edge-clamped shape functions) and Lemma 5.3 (two-edges-clamped shape functions) can be extended to the constructions of the degree of freedom linear functionals for any order $n$.

For example, suppose the 25 degree of freedom linear functionals on $Q_4(\bar{R}) = \text{span}\{\xi^i \eta^j | 0 \leq i, j \leq 4\}$ are as shown in Fig. 6 and $f = \sum_{i=0}^{4} \sum_{j=0}^{4} b_{ij} \xi^i \eta^j \in Q_4(\bar{R})$. Suppose the degree of freedom linear functionals $N_k$ on $Q_4(\bar{R})$ are defined by

$$N_k(f) = f(0, (k - 1)/4), \quad k = 1, 2, 3, 4, 5; \quad N_k(f) = f(1/3, (k - 6)/4), \quad k = 6, 7, 8, 9, 10,$$
$$N_k(f) = f(2/3, (k - 11)/4), \quad k = 11, 12, 13, 14, 15; \quad N_k(f) = f(1, (k - 16)/4), \quad k = 16, 17, 18, 19, 20, \quad (45)$$
$$N_k(f) = \partial f / \partial \xi(0, (k - 21)/4), \quad k = 21, 22, 23, 24, 25$$
Then we have the following:

\[ N_k(f) = 0 \text{ for } k = 1, 2, 3, 4, 5 \text{ and } N_k(f) = 0 \text{ for } k = 21, 22, 23, 24, 25 \text{ imply} \]

\[ b_{0,j} = b_{1,j} = 0, \text{ for } j = 0, 1, 2, 3, 4. \]  

(46)

Hence,

\[ f(\xi, \eta) = \sum_{j=0}^{4} \sum_{i=2}^{4} b_{i,j} \xi^i \eta^j. \]

(47)

Then we have the following:

\[ N_k(f) = 0 \text{ for } k = 6, 7, 8, 9, 10, \ N_k(f) = 0 \text{ for } k = 11, 12, 13, 14, 15, \text{ and } N_k(f) = 0 \text{ for } k = 16, 17, 18, 19, 20, \text{ imply that} \]

\[ \begin{bmatrix}
    b_{2,0} & b_{3,0} & b_{4,0} \\
    b_{2,1} & b_{3,1} & b_{4,1} \\
    b_{2,2} & b_{3,2} & b_{4,2} \\
    b_{2,3} & b_{3,3} & b_{4,3} \\
    b_{2,4} & b_{3,4} & b_{4,4}
\end{bmatrix}
\begin{bmatrix}
    (1/3)^2 & (2/3)^2 & 1 \\
    (1/3)^3 & (2/3)^3 & 1 \\
    (1/3)^4 & (2/3)^4 & 1 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}. \]

(47)

It follows from (46) and (47) that \( f \equiv 0 \) on \( \mathbb{R}^2 \). Thus, \( N_k, k = 1, \ldots, 25 \), are basis for the dual space \( [Q_4(\hat{R})]^* \).

[B1: One-edge-clamped particle shape functions of order 4 (Fig. 6)] Let \( L_{4,k}(\xi) \) be the \( k \)-th Lagrange interpolating polynomials of degree 3 associated with nodes 0, 1/3, 2/3, 1. Let \( L^*_{5,k}(\eta) \) be the \( k \)-th Lagrange interpolating polynomial of degree 4 associated with \( \eta = 0, 1/4, 2/4, 3/4, 1 \). Since \( L_{4,k}(0) = 0 \), for \( k = 2, 3, 4 \), all four functions, \( \xi L_{4,k}(\xi), k = 2, 3, 4 \), and their \( \xi \)-derivatives are vanishing along the \( \eta \)-axis. Thus, we have

\[
\begin{aligned}
\text{For } k = 6, 7, 8, 9, 10, \quad & \hat{\phi}_k(\xi, \eta) = c_k \xi L_{4,2}(\xi) \cdot L^*_{4,5}(\eta), \\
\text{For } k = 11, 12, 13, 14, 15, \quad & \hat{\phi}_k(\xi, \eta) = c_k \xi L_{4,3}(\xi) \cdot L^*_{4,10}(\eta), \\
\text{For } k = 16, 17, 18, 19, 20, \quad & \hat{\phi}_k(\xi, \eta) = c_k \xi L_{4,4}(\xi) \cdot L^*_{4,15}(\eta),
\end{aligned}
\]

(48)

where \( c_k \) is the normalizing constant that makes \( N_k(\hat{\phi}_k) = 1 \).

[B2: Two-edges-clamped particle shape functions of order 4 (Fig. 7)]

Suppose the degree of freedom linear functional \( N_k \) on \( Q_4(\hat{R}) = \text{span}\{\xi^i \eta^j \mid 0 \leq i, j \leq 4\} \) are defined by

\[
\begin{aligned}
N_k(f) = f(0, (k - 1)/4), k = 1, 2, 3, 4, 5; \quad & N_j(f) = f(1/3, (k - 6)/4), k = 6, 7, 9, 10, \\
N_k(f) = f(2/3, (k - 11)/4), k = 11, 12, 14, 15; \quad & N_j(f) = f(1, (k - 16)/4), k = 16, 17, 19, 20, \\
N_k(f) = \partial f / \partial \xi(0, (k - 21)/4), k = 21, 22, 23, 24, 25, \quad & N_k(f) = \partial f / \partial \eta(1/3, 0), N_j(f) = \partial f / \partial \eta(2/3, 0), \\
N_k(f) = \partial f / \partial \eta(1/3, 0), N_j(f) = \partial f / \partial \eta(2/3, 0), \quad & N_j(f) = \partial f / \partial \eta(1, 0).
\end{aligned}
\]

(49)

For \( k = 2, 3, 4 \), let \( L^*_{4,k}(\eta) \) be the \( k \)-th Lagrange interpolating polynomial of degree 3 associated with four nodes 0, 1/4, 3/4, 1. Then \( L^*_{4,k}(0) = 0 \), for \( k = 2, 3, 4 \). Hence three functions \( \eta L^*_{4,j}(\eta), j = 2, 3, 4 \), and their \( \eta \)-derivatives are vanishing along the \( \xi \)-axis. Thus, we have

\[
\begin{aligned}
\text{For } k = 7, 9, 10, \quad & \hat{\phi}_k = c_k \xi L_{4,2}(\xi) \cdot \eta L^*_{4,j}(\eta), \quad j = 2, 3, 4, \text{ respectively; } \\
\text{For } k = 12, 14, 15, \quad & \hat{\phi}_k = c_k \xi L_{4,3}(\xi) \cdot \eta L^*_{4,j}(\eta), \quad j = 2, 3, 4, \text{ respectively; } \\
\text{For } k = 17, 19, 20, \quad & \hat{\phi}_k = c_k \xi L_{4,4}(\xi) \cdot \eta L^*_{4,j}(\eta), \quad j = 2, 3, 4, \text{ respectively.}
\end{aligned}
\]

(50)

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where $c_k$ is the normalizing constant that makes $N_k(\phi_k) = 1$.

We define the degree of freedom linear functionals on $Q_0(\hat{R})$ in a similar manner. Then we have the following Lagrange particle shape functions satisfying the clamped boundary conditions on one or two sides of the reference rectangle $\hat{R}$.

[C1: One-edge-clamped particle shape functions of order 6]

Let $L_{6,k}(\xi)$ be the $k$-th Lagrange interpolating polynomials of degree 5 associated with nodes $0, 1/5, 2/5, 3/5, 4/5, 1$. Let $L_{6,k}^*(\eta)$ be the $k$-th Lagrange interpolating polynomial of degree 6 associated with $\eta = 0, 1/6, 2/6, 3/6, 4/6, 5/6, 1$. Then the following 35 particle shape functions satisfy the clamped boundary condition along the $\eta$-axis:

\[
\left\{ \begin{array}{l}
\phi_k(\xi, \eta) = c_k\xi L_{6,2}(\xi) \cdot L_{7,k-1}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,3}(\xi) \cdot L_{7,k-14}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,4}(\xi) \cdot L_{7,k-21}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,5}(\xi) \cdot L_{7,k-28}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,6}(\xi) \cdot L_{7,k-35}^*(\eta),
\end{array} \right. \tag{51}
\]

where $c_k$ is the normalizing constant that makes $N_k(\phi_k) = 1$.

[C2: Two-edge-clamped particle shape functions of order 6]

Let $L_{6,k}(\xi)$ be the $k$-th Lagrange interpolating polynomial of degree 5 associated with nodes $0, 1/5, 2/5, 3/5, 4/5, 1$. Let $L_{6,k}^*(\eta)$ be the $k$-th Lagrange interpolating polynomial of degree 5 associated with $\eta = 0, 1/6, 2/6, 3/6, 4/6, 5/6, 1$. Then the following 25 particle shape functions satisfy the clamped boundary condition along the $\xi$-axis as well as the $\eta$-axis:

\[
\left\{ \begin{array}{l}
\phi_k(\xi, \eta) = c_k\xi L_{6,2}(\xi) \cdot \eta L_{6,j}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,3}(\xi) \cdot \eta L_{6,j}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,4}(\xi) \cdot \eta L_{6,j}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,5}(\xi) \cdot \eta L_{6,j}^*(\eta), \\
\phi_k(\xi, \eta) = c_k\xi L_{6,6}(\xi) \cdot \eta L_{6,j}^*(\eta),
\end{array} \right. \tag{52}
\]

where $c_k$ is the normalizing constant that makes $N_k(\phi_k) = 1$.

6 Numerical examples

6.1 Rectangular plates

In this subsection, to demonstrate the effectiveness of the proposed meshfree method (RPPM), we calculate reliable true deflection coefficients and the true strain energy for a various forms of simply supported rectangular plates. In this subsection, we use $\phi^R_{\gamma}(\frac{x+\delta}{2\delta})$ in Eqn.(10) with $\delta = 0.05$ for the construction of generalized product PU functions so that their first order derivatives are continuous.

6.1.1 Simply supported rectangular plate

Let $\Omega = [-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$ be the rectangular plate, where $b = ar$ for some $r \geq 1$. 

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[A: Plate with uniform load] Suppose the rectangular plate is uniformly loaded by \( p(x, y) = p_0 \) for all \((x, y) \in \Omega \). By M. Lévy’s method, the deflection \( w(x, y) \) of the midplane \( \Omega \) can be expressed in the following single infinite series ([39]):

\[
w = \frac{4p_0 a^4}{\pi^5 D} \sum_{m=1}^{\infty} \frac{1}{m^5} \left( 1 - \frac{\alpha_m \tanh \alpha_m + 2}{2 \cosh \alpha_m} \cosh \frac{2\alpha_m y}{b} + \frac{\alpha_m}{2 \cosh \alpha_m} \frac{2y}{b} \sinh \frac{2\alpha_m y}{b} \right) \sin \frac{m\pi (x + \frac{a}{2})}{a},
\]

where \( \alpha_m = \frac{m\pi b}{2a} \). Here, the maximum deflection occurs at the center point \((0, 0)\) and \( w_{\text{max}} = \frac{4p_0 a^4}{\pi^5 D} \sum_{m=1}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \left( 1 - \frac{\alpha_m \tanh \alpha_m + 2}{2 \cosh \alpha_m} \right) \).

From (54), the deflection coefficient \( \beta_{\text{UL}} := \frac{w_{\text{max}} D}{p_0 a^4} \) can be written as

\[
\beta_{\text{UL}} = \frac{4}{\pi^5} \sum_{m=1}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \left( 1 - \frac{\alpha_m \tanh \alpha_m + 2}{2 \cosh \alpha_m} \right)
\]

Let \( \beta_{\text{UL}}^N \) be the \( N \)-th partial sum of the infinite series (55). Then for each positive integer \( N \), the remainder is estimated as follows:

\[
|\beta_{\text{UL}} - \beta_{\text{UL}}^N| \leq \frac{4}{\pi^5} \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^5} \left| 1 - \frac{\alpha_{2m-1} \tanh \alpha_{2m-1} + 2}{2 \cosh \alpha_{2m-1}} \right|
\]

\[
\leq \frac{4}{\pi^5} \int_N^{\infty} \frac{1}{(2x-1)^5} dx \leq \frac{1}{2\pi^5 (2N-1)^4}.
\]

[B: Plate with point load] When a rectangular plate is loaded at center point only with \( p(0, 0) = p_0 \), the deflection of the midplane can be expressed in the following infinite series ([39]):

\[
w = \frac{p_0 a^2}{2\pi^3 D} \sum_{m=1}^{\infty} \left( 1 + \alpha_m \tanh \alpha_m \right) \sinh \frac{(b-2y)\alpha_m}{b} - \frac{(b-2y)\alpha_m}{b} \cosh \frac{(b-2y)\alpha_m}{b}
\]

\[
\left( \frac{\sin \frac{m\pi}{2} \sin \frac{m\pi (x + \frac{a}{2})}{a}}{m^3 \cosh \alpha_m} \right), \quad \alpha_m = \frac{m\pi b}{2a}.
\]

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Hence, we have

\[ w_{\text{max}} = w(0, 0) = \frac{p_0 a^2}{2 \pi^3 D} \sum_{m=1}^{\infty} \left( (1 + \alpha_m \tanh \alpha_m) \sinh \alpha_m - \alpha_m \cosh \alpha_m \right) \frac{\sin^2 \frac{m \pi}{2}}{m^3 \cosh \alpha_m} \]

\[ = \frac{p_0 a^2}{2 \pi^3 D} \sum_{m=1}^{\infty} \frac{1}{m^3} (1 + \alpha_m \tanh \alpha_m) \sinh \alpha_m - \alpha_m \cosh \alpha_m \]

\[ = \frac{p_0 a^2}{2 \pi^3 D} \sum_{m=1}^{\infty} \frac{1}{m^3} \left( \tanh \alpha_m - \frac{\alpha_m}{\cosh^2 \alpha_m} \right). \]

The deflection coefficient \( \beta_{\text{PL}} := \frac{w_{\text{max}} D}{p_0 a^2} \) now can be expressed in the form

\[ \beta_{\text{PL}} = \frac{1}{2 \pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \left( \tanh \alpha_{2m-1} - \frac{\alpha_{2m-1}}{\cosh^2 \alpha_{2m-1}} \right). \] (58)

The remainder after the \( N \)-th partial sum \( \beta_{\text{PL}}^N \) of the series (58) is estimated as follows:

\[ | \beta_{\text{UL}} - \beta_{\text{UL}}^N | \leq \frac{1}{2 \pi^3} \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^3} \left| \tanh \alpha_{2m-1} - \frac{\alpha_{2m-1}}{\cosh^2 \alpha_{2m-1}} \right| \leq \frac{1}{2 \pi^3} \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^3} \]

\[ \leq \frac{1}{2 \pi^3} \int_N^{\infty} \frac{1}{(2x-1)^3} dx \]

\[ = \frac{1}{8 \pi^3 (2N-1)^2}. \] (59)

[C: Estimated true deflections at (0, 0)] Using (56) and (59), we determine \( N \) such that the \( N \)-th partial sums differ from the true values within \( 5.0 \times 10^{-13} \) (with double precision calculation in a 32 bit computer). For this purpose, we solve the following inequalities for \( N \):

\[ \frac{1}{2 \pi^5 (2N-1)^4} \leq 5.0 \times 10^{-13} \quad \text{and} \quad \frac{1}{8 \pi^3 (2N-1)^2} \leq 5.0 \times 10^{-13}, \]

from which we get \( N = 121 \) and \( N = 44898 \), respectively.

Table 3 shows approximated deflection coefficients and their effective digits, which means that they are exactly matched with those of the true deflection coefficients up to that many digits. In the subsequent subsection, we use the numbers in the third and the fifth columns of Table 3 as the true solutions.

[D: The computed true strain energy] Let us denote the infinite series (53) for a square plate with uniform load and the infinite series (57) for a square plate with point load by \( w_{\text{UL}} \) and \( w_{\text{PL}} \), respectively. Using the first 200 terms (the partial sums \( S_{200}^{\text{UL}}(x, y) \) and \( S_{200}^{\text{PL}}(x, y) \) ) of each series, we compute the energy of \( w_{\text{UL}} \) and \( w_{\text{PL}} \), respectively, as follows:

\[ U(w_{\text{UL}}) = \frac{1}{2} B(w_{\text{UL}}, w_{\text{UL}}) \approx \frac{1}{2} B(S_{200}^{\text{UL}}(x, y), S_{200}^{\text{UL}}(x, y)) = 4.337005274865745, \] (60)

\[ U(w_{\text{PL}}) = \frac{1}{2} B(w_{\text{PL}}, w_{\text{PL}}) \approx \frac{1}{2} B(S_{200}^{\text{PL}}(x, y), S_{200}^{\text{PL}}(x, y)) = 228.0256112890886. \] (61)
Table 3: For various ratios \( b/a \), we list the 121-th partial sum of the series (55) and the 44898-th partial sum of the infinite series (58) and their effective digits (the digits that are exactly matched with those of the true deflection coefficients \( \beta_{UL} \) and \( \beta_{PL} \), respectively).

<table>
<thead>
<tr>
<th>( b/a )</th>
<th>Uniform load ( \beta_{UL} )</th>
<th>effective digits</th>
<th>Point load ( \beta_{PL} )</th>
<th>effective digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.00406235266068292</td>
<td>0.00406235266</td>
<td>0.01160083977171165</td>
<td>0.01160083977</td>
</tr>
<tr>
<td>1.2</td>
<td>0.00565053001076724</td>
<td>0.00565053001</td>
<td>0.0135598684715661</td>
<td>0.0135598684</td>
</tr>
<tr>
<td>1.4</td>
<td>0.00708491657891541</td>
<td>0.00708491657</td>
<td>0.0148667151453292</td>
<td>0.0148667151</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0083081160930651</td>
<td>0.00830811600</td>
<td>0.0157009619174496</td>
<td>0.0157009619</td>
</tr>
<tr>
<td>1.8</td>
<td>0.00931591418277626</td>
<td>0.00931591418</td>
<td>0.01621461935038624</td>
<td>0.0162146193</td>
</tr>
<tr>
<td>2.0</td>
<td>0.01012866305521370</td>
<td>0.01012866305</td>
<td>0.01652394919176146</td>
<td>0.0165239491</td>
</tr>
</tbody>
</table>

In subsequent subsection, we compute the relative errors in energy norm by applying these computed true energies to the following formula:

\[
\| \text{Rel Err} \|_{\text{Eng}} = \left[ \frac{B(u_{\text{true}},u_{\text{true}}) - B(u_{\text{app}},u_{\text{app}})}{B(u_{\text{true}},u_{\text{true}})} \right]^{1/2}.
\] (62)

The relative error in maximum norm \( \| \text{Rel Err} \|_{\infty} \) is defined similarly.

It is quite involved to express the displacement \( w(x,y) \) of a clamped rectangular plate in a single infinite series. Thus, we do not compute the true energy of a clamped square plate in this paper. However, Example 6.3 shows that our method also effectively handles clamped plate problems.

6.1.2 Examples

**Example 6.1.** With material constants \( \nu = 0.3, E = 10^9, a = 0.6, d = 0.001, p = 100.0 \) (These material constants are the same as those in [15]). We test the proposed meshfree particle methods to the following cases:

A: Simply supported rectangular plate with uniform load (Table 4, Table 5);
B: Simply supported rectangular plate with point load at the center (Table 4, Table 6);
C: Clamped rectangular plate with uniform load (Table 7);
D: Clamped rectangular plate with point load at the center (Table 8).

[A: Relative errors in energy norm] In Table 4, the relative errors in energy norm are computed by using the computed true solutions given by (60) and (61). The analytic solution for the displacement \( w_{PL} \) of a simply supported rectangular plate with point load has a singularity (of type \( r^2 \log r \)) at the center point. Thus if all other data related to the thin plate with point load are smooth, we have

\[ w_{PL} \in H^2(\Omega), \]

whereas if the plate is uniformly loaded, then the displacement \( w_{UL} \) is highly regular. The effect of the regularity of \( w_{UL} \) and \( w_{PL} \) to the accuracy of the computed solutions are well reflected
in Table 4 and Fig. 8, in which the relative errors in energy norm versus the RPP orders are plotted on log × log-scale.

The asymptotic rates of convergence in the energy norm for errors against DOF (i.e. the increment in $\log(\|\text{Rel Err}\|_{\text{eng}})$ over the increment in $\log(\text{DOF})$) are about 0.5 and 3.5 for the simply supported square plate with point load and with uniform load, respectively. And the asymptotic rates of convergence in the energy norm for errors against RPP orders are about 1.0 and 7.0, respectively.

Table 4: Relative errors in energy norm (%) for simply supported square plate.

<table>
<thead>
<tr>
<th>rpp order</th>
<th>Uniform Load</th>
<th>Point Load</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$|\text{Rel Err}|_{\text{eng}}$</td>
<td>$|\text{Rel Err}|_{\text{eng}}$</td>
</tr>
<tr>
<td>$k$</td>
<td>DOF</td>
<td>(%)</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>9.2296E-00</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
<td>1.9197E-00</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>4.0845E-01</td>
</tr>
<tr>
<td>6</td>
<td>144</td>
<td>8.6116E-02</td>
</tr>
<tr>
<td>7</td>
<td>196</td>
<td>2.8965E-02</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>1.1261E-02</td>
</tr>
<tr>
<td>$\infty$ (True)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 8: Convergence of the proposed method for simply supported square plate in energy norm.

[B: Relative errors in maximum norm] In the following, we use the following notations:

- $\beta^{UL} = \frac{w_{\max}D}{P a^4}$ for uniform load and $\beta^{PL} = \frac{w_{\max}D}{P a^2}$ for point load, where $P$ is load, $w_{\max}$
is the vertical displacement at the center point, \( a \) and \( b \) are the two side lengths of a rectangular plate shown Fig. 10. “DOF” stands for the degrees of freedom, that is the number of particles employed. However, those particles located along the boundary with essential boundary conditions of Table 2 were not counted into DOF.

- the results in the rows “\( \beta_{RPP\;4} \)”, “\( \beta_{RPP\;5} \)” and “\( \beta_{RPP\;6} \)” are those obtained by our method with use of particle shape functions of RPP order 4, RPP order 5 and RPP order 6, respectively. The results in the row “\( \beta_{Liu} \)” are those in ([14],[15]) and the results in the row “\( \beta_{Timo} \)” are the solutions in Timoshenko([39]). “\( \beta^{UL}_{Taylor} \)” are results in ([38]).

From results in Tables 5, 6, 7, and 8, we observe the following

1. Tables 6 and 8 show that even though our method uses much smaller number of particles (144 in \( \beta_{RPP\;6} \) and 256 in \( \beta_{Liu} \)), our method yields better results than the moving least squares (MLS) method employed in ([14]). Actually, the results in the row “\( \beta_{RPP\;4} \)” that use 36 particles are similar to those number in “\( \beta_{Liu} \)” that use 256(16 \times 16) uniformly spaced particles.

2. Implementing RPP shape functions satisfying clamped BC and constructing generalized product PU functions for the proposed meshfree particle method is simple.

3. Tables 5 and 7 show that in the case of uniform load, the results in the row “\( \beta_{RPP\;4} \)” show that our method using 64 particles for simply supported BC (36 particles for clamped BC) yields already enough accuracy. In the case of uniform load, our method yields almost the same as the computed true solution.
4. Because of the high regularity of the displacement of rectangular plate with uniform load, the numerical solutions in the Tables 5 and 7 are better than those in Tables 6 and 8 (for the plate with point load).

5. The results in the row “True” are the numbers in the Table 3. Actually, they agree with the true solution at least 10 effective digits after decimal point. Fig. 9 shows the convergence rates of the displacement of a square plate in maximum norm that are effected by the regularity of the true displacement.

Table 5: $\beta^{UL}$ for various ratios $b/a$ for a simply supported rectangular plate with uniform load $P = 100$.

<table>
<thead>
<tr>
<th>DOF</th>
<th>$b/a = 1.0$</th>
<th>$b/a = 1.2$</th>
<th>$b/a = 1.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^{UL}_{RPP 4}$</td>
<td>64 0.004061654132985</td>
<td>0.005648661941758</td>
<td>0.007080361684012</td>
</tr>
<tr>
<td>$\beta^{UL}_{RPP 6}$</td>
<td>144 0.004062354883286</td>
<td>0.005650543751673</td>
<td>0.007084965007638</td>
</tr>
<tr>
<td>$\beta^{UL}_{RPP 8}$</td>
<td>256 0.004062352582624</td>
<td>0.005650592716976</td>
<td>0.00708495122955</td>
</tr>
<tr>
<td>$\beta^{UL}_{Timo}$</td>
<td>0.00406</td>
<td>0.00564</td>
<td>0.00705</td>
</tr>
</tbody>
</table>

Table 5: $\beta^{UL}$ for various ratios $b/a$ for a simply supported rectangular plate with uniform load $P = 100$.

<table>
<thead>
<tr>
<th>DOF</th>
<th>$b/a = 1.6$</th>
<th>$b/a = 1.8$</th>
<th>$b/a = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^{UL}_{RPP 4}$</td>
<td>64 0.008290900469166</td>
<td>0.009300429398380</td>
<td>0.01014894965213</td>
</tr>
<tr>
<td>$\beta^{UL}_{RPP 6}$</td>
<td>144 0.008308218723322</td>
<td>0.009316079093987</td>
<td>0.01012880486526</td>
</tr>
<tr>
<td>$\beta^{UL}_{RPP 8}$</td>
<td>256 0.008308113118978</td>
<td>0.009315909184649</td>
<td>0.01012865691790</td>
</tr>
<tr>
<td>$\beta^{UL}_{Timo}$</td>
<td>0.00830</td>
<td>0.00931</td>
<td>0.01013</td>
</tr>
</tbody>
</table>

In the next two examples, the closed form-exact solutions are known.

6.2 Triangular and circular plates

For Example 6.2, we use $\phi_{\alpha R}^R(x+\delta)$ in Eqn.(10) with $\delta = 0.1$ for the construction of generalized product PU functions.

Example 6.2. (Simply supported triangular plate) For simply supported equilateral triangular plates under uniform lateral loads, a closed form solution has been obtained by Woinowsky-Krieger([39],[41]) in the following form:

$$w(x,y) = \frac{P}{64Da} \left[ x^3 - x^2a - 3xy^2 - y^2a + \frac{4}{27}a^3 \right] \left[ \frac{4}{9}a^2 - x^2 - y^2 \right].$$

(63)

The plate shown in Fig. 10 is an equilateral triangle with height $a$ and the vertex of the plate on the $x$-axis is $(2a/3,0)$, and $P$ is the uniform load constant.
Figure 10: (a) Rectangular plate and partition into four patches. $Q_j^{fft}$ is the flat-top part of the supp$\Psi_j^P$ of the generalized product PU function. (b) Simply supported triangular plate and (c) partition into three patches.
Table 6: $\beta_{UL}$ for various ratios $b/a$ for a simply supported rectangular plate with point load $P = 100$ at the origin.

<table>
<thead>
<tr>
<th></th>
<th>DOF</th>
<th>$b/a = 1.0$</th>
<th>$b/a = 1.2$</th>
<th>$b/a = 1.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{RPP_4}$</td>
<td>64</td>
<td>0.011541868297442</td>
<td>0.013484365037024</td>
<td>0.014776660124860</td>
</tr>
<tr>
<td>$\beta_{RPP_6}$</td>
<td>144</td>
<td>0.011574431960126</td>
<td>0.013526455087453</td>
<td>0.014834541742966</td>
</tr>
<tr>
<td>$\beta_{RPP_8}$</td>
<td>256</td>
<td>0.011586369702061</td>
<td>0.013540579388621</td>
<td>0.014849773345839</td>
</tr>
<tr>
<td>$\beta_{Lin}$</td>
<td>256</td>
<td>0.011574431960126</td>
<td>0.013526455087453</td>
<td>0.014834541742966</td>
</tr>
<tr>
<td>$\beta_{Timo}$</td>
<td>256</td>
<td>0.011586369702061</td>
<td>0.013540579388621</td>
<td>0.014849773345839</td>
</tr>
<tr>
<td>True</td>
<td>$\infty$</td>
<td>0.011600839771712</td>
<td>0.013555986847157</td>
<td>0.014866715145323</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>DOF</th>
<th>$b/a = 1.6$</th>
<th>$b/a = 1.8$</th>
<th>$b/a = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{RPP_4}$</td>
<td>64</td>
<td>0.015585695738181</td>
<td>0.016066259768443</td>
<td>0.01633541110802</td>
</tr>
<tr>
<td>$\beta_{RPP_6}$</td>
<td>144</td>
<td>0.015666439457326</td>
<td>0.016177691470258</td>
<td>0.016484311512675</td>
</tr>
<tr>
<td>$\beta_{RPP_8}$</td>
<td>256</td>
<td>0.015682034873019</td>
<td>0.016193642951648</td>
<td>0.01650107962984</td>
</tr>
<tr>
<td>$\beta_{Lin}$</td>
<td>256</td>
<td>0.015585695738181</td>
<td>0.016066259768443</td>
<td>0.01633541110802</td>
</tr>
<tr>
<td>$\beta_{Timo}$</td>
<td>256</td>
<td>0.015666439457326</td>
<td>0.016177691470258</td>
<td>0.016484311512675</td>
</tr>
<tr>
<td>True</td>
<td>$\infty$</td>
<td>0.015700961941745</td>
<td>0.016214619350386</td>
<td>0.016523949191761</td>
</tr>
</tbody>
</table>

With $P = 10$ and $a = 1$, the proposed methods are tested with respect to various plate thickness and various RPP order of local approximation functions.

The true displacement (63) for this triangular plate is a polynomial of degree 5. Moreover, the sum of the degree in the $x$-variable and the degree in the $y$-variable of each term in the true solution is less than or equal to 5. Hence, Lemma 3.1 implies that the polynomial (63) can be generated by the RPP shape functions of order 5. Thus, our method with RPP order 6 almost captures the exact solution as shown in Table 9.

Finally, we consider a thin plate with curved sides. Under uniform lateral loads ($P(x,y) = P$ for all $(x,y) \in \Omega$), it is shown in [37] that a closed form solution of a clamped elliptical plate with intersections $\pm a$ and $\pm b$ along the $x$-axis and the $y$-axis, respectively, is given by the following form:

$$w = \frac{Pa^4b^4}{8D[3(a^2+b^2)^2-(2ab)^2]} \left[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right]^2. \quad (64)$$

In the following, we test the proposed method to the case where $a = b = 1$ in the Eqn. (64).

**Example 6.3.** (Clamped Circular Plate) For clamped circular plates, the particles and patches are as shown in Fig. 11. With $P = 1, E = 1000, \nu = 0.25$ and the thickness, $d = 0.05$, an application of the proposed method to the patches shown in Fig. 11 yields the relative errors in energy norm as well as maximum norm shown in Table 10.

Planting particles and constructing particle shape functions for the circular plate are as follows:
Table 7: Values of $\beta^{UL} \times 1000$ for different ratios of $b/a$ corresponding to a uniform load of $P = 100$ for a clamped plate.

<table>
<thead>
<tr>
<th></th>
<th>DOF</th>
<th>$b/a = 1.0$</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^{UL}_{RPP 4}$</td>
<td>36</td>
<td>1.26032</td>
<td>1.71577</td>
<td>2.05142</td>
<td>2.27189</td>
<td>2.40311</td>
<td>2.47162</td>
</tr>
<tr>
<td>$\beta^{UL}_{RPP 5}$</td>
<td>64</td>
<td>1.26457</td>
<td>1.72409</td>
<td>2.06759</td>
<td>2.30007</td>
<td>2.44763</td>
<td>2.53683</td>
</tr>
<tr>
<td>$\beta^{UL}_{RPP 6}$</td>
<td>100</td>
<td>1.26532</td>
<td>1.72490</td>
<td>2.06823</td>
<td>2.30013</td>
<td>2.44639</td>
<td>2.53320</td>
</tr>
<tr>
<td>$\beta^{UL}_{Timo}$</td>
<td>1.26</td>
<td>1.72</td>
<td>2.07</td>
<td>2.30</td>
<td>2.45</td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td>$\beta^{UL}_{Taylor}$ ([38])</td>
<td>1.26532</td>
<td>1.72487</td>
<td>2.06814</td>
<td>2.29997</td>
<td>2.44616</td>
<td>2.53297</td>
<td></td>
</tr>
</tbody>
</table>

Table 8: Values of $\beta^{PL} \times 1000$ for various ratios of $b/a$ corresponding to a point load of $P = 100$ at the origin for a clamped plate.

<table>
<thead>
<tr>
<th></th>
<th>DOF</th>
<th>$b/a = 1.0$</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^{PL}_{RPP 4}$</td>
<td>36</td>
<td>5.53147</td>
<td>6.37482</td>
<td>6.79340</td>
<td>6.95896</td>
<td>6.99601</td>
<td>6.97164</td>
</tr>
<tr>
<td>$\beta^{PL}_{RPP 5}$</td>
<td>64</td>
<td>5.57200</td>
<td>6.43307</td>
<td>6.87598</td>
<td>7.07175</td>
<td>7.14521</td>
<td>7.16419</td>
</tr>
<tr>
<td>$\beta^{PL}_{RPP 6}$</td>
<td>100</td>
<td>5.58560</td>
<td>6.44588</td>
<td>6.88885</td>
<td>7.08653</td>
<td>7.16427</td>
<td>7.19037</td>
</tr>
<tr>
<td>$\beta^{PL}_{Liu}$ ([15])</td>
<td>196</td>
<td>5.52</td>
<td>6.37</td>
<td>6.80</td>
<td>6.98</td>
<td>7.03</td>
<td>7.04</td>
</tr>
<tr>
<td>$\beta^{PL}_{Timo}$ ([39])</td>
<td>5.60</td>
<td>6.47</td>
<td>6.91</td>
<td>7.12</td>
<td>7.20</td>
<td>7.22</td>
<td></td>
</tr>
</tbody>
</table>

1. The circular plate is partitioned into the eight curved rectangular patches and one disk patch as shown in Fig. 11. On each of the eight rectangular patches shown in the top part of Fig. 11, we construct particles and particle shape functions by using Lemma 5.2 and degree of freedom functions depicted in Fig. 7. Through the patch mapping, $T(r, \theta) = (r \cos \theta, r \sin \theta)$, the particles and the particle shape functions are planted on the annulus region shown in the bottom part of Fig. 11.

2. Partition of unity functions and particle shape functions on the circular plate also constructed through the patch mapping $T$. Particle shape functions for the disk patch are constructed by multiplying the tensor product of Lagrange interpolation functions by partition of unity function with flat-top as shown in Fig. 12. Let us note that these particle shape functions have no Kronecker delta property at those particles that go outside the support of PU function.

3. By Lemma 5.2, all particle shape functions corresponding to particles planted in the curved rectangular patches and the disk patch and their normal derivatives are vanishing along the boundary of the circular plate. That is, all of particle shape functions satisfy the clamped boundary conditions.
Table 9: Simply supported triangular plate with thickness $d = 0.005$ (top) and $d = 0.05$ (bottom).

<table>
<thead>
<tr>
<th>RPP order</th>
<th>DOF</th>
<th>$|\text{Rel Err}|_\infty$</th>
<th>$|\text{Rel Err}|_{\text{Eng}}$</th>
<th>Computed Eng</th>
<th>True Eng</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>9</td>
<td>6.439E-02</td>
<td>5.914E-01</td>
<td>6.7034730963011E+02</td>
<td>1.0309826235529E+03</td>
</tr>
<tr>
<td>4</td>
<td>38</td>
<td>4.939E-03</td>
<td>9.091E-02</td>
<td>1.0224620764897E+03</td>
<td>1.0309826235529E+03</td>
</tr>
<tr>
<td>6</td>
<td>87</td>
<td>7.128E-13</td>
<td>1.714E-06</td>
<td>1.03098262355499E+03</td>
<td>1.0309826235529E+03</td>
</tr>
</tbody>
</table>

Table 10: Clamped circular plate with thickness $d = 0.05$.

<table>
<thead>
<tr>
<th>RPP order</th>
<th>DOF</th>
<th>$|\text{Rel Err}|_\infty$</th>
<th>$|\text{Rel Err}|_{\text{Eng}}$</th>
<th>Computed Eng</th>
<th>True Eng</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>80</td>
<td>1.099E-01</td>
<td>1.968E-01</td>
<td>7.0778942998948E-01</td>
<td>7.3631077818511E-01</td>
</tr>
<tr>
<td>4</td>
<td>145</td>
<td>1.723E-13</td>
<td>3.306E-07</td>
<td>7.3631077818502E-01</td>
<td>7.3631077818511E-01</td>
</tr>
</tbody>
</table>

Figure 11: Diagram of eight curved rectangular patches and one disk patch on the circular plate. Particles on the circular plate are constructed by the patch mapping $T(r, \theta) = (r \cos \theta, r \sin \theta)$ Here $\delta_\theta = \pi/12$, $\delta_r = 0.1$ and the radius of the inner circle (solid line) is 0.6.
Figure 12: Particles on the disk patch of the circular plate. RPP shape functions are tensor product of Lagrange interpolation functions. Some particles are outside the disk patch.

7 Concluding remarks

The particle shape functions in the proposed method (RPPM) can be constructed to be smooth up to any desired order. Moreover, the approximation functions used in this paper satisfy the boundary conditions arising in the plate problems.

Numerical examples show that our method can effectively handle the thin plates (Kirchhoff plates) whenever they are of convex polygonal shape. In order to get highly accurate numerical solutions for plates with a curved boundary, much care should be exercised in the construction of particle shape functions as well as partition of unity functions as shown in Example 3.

In this paper, we considered the numerical solutions of the fourth order partial differential equations in convex domains. The plates with corners or cracks will be considered in other papers.

A Proof of Lemma 5.1:

Proof. (i) By Lemma 3.1.4 of ([6]), it suffices to show that for a given \( f \in P_3(\hat{T}) \) with \( N_i(f) = 0 \) for \( i = 1, \cdots, 10 \), we have \( f \equiv 0 \).

Now, for some constants \( a_{ij} \), \( f(\xi, \eta) \) can be written as follows:

\[
 f(\xi, \eta) = \sum_{0 \leq i, j \leq 3, 0 \leq i + j \leq 3} a_{ij} \xi^i \eta^j \tag{65}
\]

and hence \( f(0, \eta) = \sum_{j=0}^{3} a_{0,j} \eta^j \). Since \( N_k(f) = 0 \) for \( k = 1, 2, 3, 4 \), we have

\[
 a_{0,j} = 0, \quad j = 0, 1, 2, 3. \tag{66}
\]

On the other hand, we have

\[
 \frac{\partial f}{\partial \xi}(0, \eta) = \left[ \sum_{1 \leq i, j \leq 3, 0 \leq i+j \leq 3} a_{i,j}(i \xi^{i-1})(\eta^j) \right](0, \eta) = \sum_{j=0}^{2} a_{1,j} \eta^j, \tag{67}
\]

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Since $N_j(f) = 0$, for $j = 8, 9, 10$, we have
\[ a_{1j} = 0, \text{ for } j = 0, 1, 2 \] (68)

From (66) and (68), we have
\[ f(\xi, \eta) = [a_{2,0}\xi^2 + a_{3,0}\xi^3] + [a_{2,1}\xi^2]\eta \] (69)

Since $N_5(f) = 0, N_7(f) = 0$,
\[ \begin{bmatrix} (1/2)^2 & (1/2)^3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{2,0} \\ a_{3,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \] (70)

Hence $f = a_{2,1}\xi^2\eta$. Finally, it follows from $N_6(f) = 0$ that $f \equiv 0$ on $\mathbb{R}^2$.

(ii) It follows from (66) that $f(0, \eta) \equiv 0$ for all $\eta \in \mathbb{R}$. Since
\[ \frac{\partial f}{\partial \eta}(0, \eta) = \sum_{0 \leq i, j \leq 3, 0 \leq i+j \leq 3} a_{i,j}(i\xi^i(j\eta^{j-1}))(0, \eta) = \sum_{j=1}^{3} a_{0,j}(j\eta^{j-1}), \]
(66) implies that $\partial f/\partial \eta$ is also identically zero along the $\eta$-axis.

Similarly, the conditions: $N_j(f) = 0$, for $j = 8, 9, 10$, imply that
\[ \frac{\partial f}{\partial \xi}(0, \eta) = \sum_{1 \leq i, j \leq 3, 0 \leq i+j \leq 3} a_{i,j}(i\xi^i)(\eta^j)(0, \eta) = \sum_{j=0}^{3} a_{1,j}(\eta^j), \] (71)
is 0 for $\eta = 0, 1/3, 1$. Hence, $\partial f/\partial \xi \equiv 0$ along the line $\xi = 0$.

\[ \square \]

B Proof of Lemma 5.2:

Proof. (i) Suppose $N_j(f) = 0, j = 1, \cdots, 16$. Now $f(\xi, \eta) \in Q_3(\hat{R})$ can be written as
\[ f(\xi, \eta) = \sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,j}\xi^i\eta^j \] (72)
and hence $f(0, \eta) = \sum_{j=0}^{3} b_{0,j}\eta^j$. Since $N_k(f) = f(0, (k-1)/3) = 0$ for $k = 1, 2, 3, 4$, we have
\[ b_{0,j} = 0, \text{ for } j = 0, 1, 2, 3. \] (73)

On the other hand, we have
\[ \frac{\partial f}{\partial \xi}(0, \eta) = \sum_{i=1}^{3} \sum_{j=0}^{3} b_{i,j}(i\xi^{i-1})(\eta^j)(0, \eta) = \sum_{j=0}^{3} b_{1,j}(\eta^j), \] (74)
Since $N_j(f) = \partial f/\partial \xi(0, (j - 13)/3) = 0$, for $j = 13, 14, 15, 16$, we have
\[ b_{1,j} = 0, \text{ for } j = 0, 1, 2, 3. \] (75)
From (73) and (75), we have

\[ f(\xi, \eta) = \sum_{j=0}^{3} (b_{2,j} \xi^2 + b_{3,j} \xi^3) \eta^j. \]  

(76)

Since \( N_k(f) = f(1/2, (k - 5)/3) = 0 \), for \( k = 5, 6, 7, 8, \) and \( N_j(f) = f(1, (k - 9)/3) = 0 \) for \( k = 9, 10, 11, 12, \) we have

\[
\begin{bmatrix}
  b_{2,0} & b_{3,0} \\
  b_{2,1} & b_{3,1} \\
  b_{2,2} & b_{3,2} \\
  b_{2,3} & b_{3,3}
\end{bmatrix}
\begin{bmatrix}
  (1/2)^2 & 1 \\
  (1/2)^3 & 1
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}.
\]  

(77)

Since the second matrix of (77) is invertible, we have

\[ b_{2,j} = b_{3,j} = 0, \text{ for } j = 0, 1, 2, 3. \]  

(78)

(73), (75), and (78) imply that \( f \equiv 0 \) on \( \mathbb{R}^2 \).

(ii) \( f(0, \eta) = [\sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,j} (i \xi^{i-1}) (\eta^j)](0, \eta) = \sum_{j=0}^{3} b_{0,j} \eta^j \). From the conditions: \( f(0, 0) = f(0, 1/3) = f(0, 2/3) = f(0, 1) = 0 \), we have

\[ b_{0,j} = 0, \text{ for } j = 0, 1, 2, 3, \]  

(79)

which implies that \( f \equiv 0 \) along the line \( \xi = 0 \).

Since \( \frac{\partial f}{\partial \xi}(0, \eta) = [\sum_{i=0}^{3} \sum_{j=0}^{3} b_{i,j} (i \xi^{i-1}) (\eta^j)](0, \eta) = \sum_{j=0}^{3} b_{1,j} (\eta^j) \), the assumptions: \( \frac{\partial f}{\partial \xi}(0, 0) = \frac{\partial f}{\partial \xi}(0, 1/3) = \frac{\partial f}{\partial \xi}(0, 2/3) = \frac{\partial f}{\partial \xi}(0, 1) = 0 \), yield

\[ b_{1,j} = 0, \text{ for } j = 0, 1, 2, 3, \]  

(80)

which shows that \( \frac{\partial f}{\partial \xi} \equiv 0 \) along the line \( \xi = 0 \).

We have \( \frac{\partial f}{\partial \eta}(0, \eta) = \sum_{j=1}^{3} b_{0,j} (j \eta^{j-1}) = b_{0,1} + 2b_{0,2} \eta + 3b_{0,3} \eta^2 = 0 \), by (79). That is, \( \frac{\partial f}{\partial \eta} \equiv 0 \) along the \( \eta \)-axis.

\[ \square \]

C  Proof of Lemma 5.3:

Proof. (i) The proof of this part is similar to that of Lemma 5.2.

(ii) \( f(0, \eta) = [\sum_{i=1}^{3} \sum_{j=0}^{3} b_{i,j} (i \xi^{i-1}) (\eta^j)](0, \eta) = \sum_{j=0}^{3} b_{0,j} (\eta^j) \). From the conditions: \( f(0, 0) = f(0, 1/3) = f(0, 2/3) = f(0, 1) = 0 \), we have

\[ b_{0,j} = 0, \text{ for } j = 0, 1, 2, 3, \]  

(81)

which shows that \( f \equiv 0 \) along the \( \eta \)-axis.

\[ \frac{\partial f}{\partial \xi}(0, \eta) = [\sum_{i=1}^{3} \sum_{j=0}^{3} b_{i,j} (i \xi^{i-1}) (\eta^j)](0, \eta) = \sum_{j=0}^{3} b_{1,j} (\eta^j). \]  

The assumptions: \( \frac{\partial f}{\partial \xi}(0, 0) = \frac{\partial f}{\partial \xi}(0, 1/3) = \frac{\partial f}{\partial \xi}(0, 2/3) = \frac{\partial f}{\partial \xi}(0, 1) = 0 \) yields

\[ b_{1,j} = 0, \text{ for } j = 0, 1, 2, 3, \]  

(82)

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which shows $\frac{\partial f}{\partial \xi}(\xi,0) = 0$ along the $\xi$-axis.

(81) implies $\frac{\partial f}{\partial \eta}(0,\eta) = \sum_{j=1}^{3} b_{0,j} (j\eta^{j-1}) = 0$ along the $\eta$-axis.

Since $b_{0,0} = b_{1,0} = 0$, we have $f(\xi,0) = \sum_{i=0}^{3} b_{i,0}(\xi^{i}) = b_{2,0}\xi^{2} + b_{3,0}\xi^{3}$. The assumptions: $f(1/2,0) = f(1,0) = 0$ implies

$$b_{i,0} = 0, \text{ for } i = 2, 3,$$

which shows $f \equiv 0$ along the $\xi$-axis.

By (81) and (82), $b_{0,1} = b_{1,1} = 0$, and hence $\frac{\partial f}{\partial \eta}(\xi,0) = \sum_{i=0}^{3} b_{i,1}(\xi^{i}) = b_{2,1}\xi^{2} + b_{3,1}\xi^{3}$. Now the assumptions: $\frac{\partial f}{\partial \eta}(1/2,0) = \frac{\partial f}{\partial \eta}(1,0) = 0$ implies

$$b_{2,1} = b_{3,1} = 0,$$

which shows $\frac{\partial f}{\partial \eta} \equiv 0$ along the $\xi$-axis.

From (82) and (83), we have $b_{1,0} = b_{2,0} = b_{3,0} = 0$ and hence $\frac{\partial f}{\partial \xi}(\xi,0) = \sum_{i=1}^{3} b_{i,0}(i\xi^{i-1}) = 0$. That is, $\frac{\partial f}{\partial \xi} \equiv 0$ along the $\xi$-axis.

\[\square\]

References


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