Increasing the Reliability of a Machine
Reduces the Period of its Work,

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Abstract

The comparison of optimal strategies in a simple stochastic replacement model
for two types of machines with identical cost characteristics when one of them is
more reliable than the other is conducted. It is proven that scheduled replacement
period for more reliable type is always less than for the less reliable one. An
example is presented when even the expected period of use of a more reliable
machine is less than the expected period for the less reliable one. Some related
problems are briefly discussed.

Key words: REPLACEMENT POLICY, RELIABILITY, MARKOV DECISION PROCESS


1 Introduction

At first glance the title of this paper seems absurd. The term ”more reliable” is usually
understood to mean ”capable of working a longer period”. Of course, this is quite true if
a machine (unit, system, equipment, item) is replaced only for purely technical reasons, such as failure or the deterioration of parameters beyond acceptable limits. But in most cases the replacement of a machine is related to cost considerations. So even a functioning machine can be replaced after some period of time, called the replacement period, due to increasing operating costs. In the stochastic case, when a machine is subjected to random failure, the actual period of its use can be even less than the scheduled replacement period if repairs are impossible and a machine has to be replaced immediately because of failure. In such situations, *apriori*, it is not clear how a change in the reliability of a machine influences its optimal replacement period and correspondingly its actual period of use. Consider the following simplified situation. There are two automobile companies, say Mord and Fotors, producing cars and there are two families, say Mitchells and Franklins, buying these cars. The customs of the two families are such that M’s always buy the cars M and F’s always buy the cars F. The model can also be interpreted to cover the case of two different machines operating in two different industries. The cars of both companies are subjected to random failure. If a car did not fail after $n$ periods of time $n = 1, 2, \ldots$, then a family, say, M has two options: to buy a new car from its firm at cost $C^n_M$, or to use the current car in the next period at operating cost $c^n_M$, where $n$ is the number of periods of time elapsed since the last purchase of a new car. Thus we assume for simplicity that the cost of a new car and its operating costs are the same for each new generation of cars (and for each firm). In the event of failure, a car has to be replaced by a new one immediately. The last assumption is not too unrealistic, since the possibilities of minor failures and possible repairs can be included in the operating costs. The goal of each family is to minimize its total expected discounted cost over an infinite time interval. The case of average cost is considered also. Under the plausible assumption that $c^n_M$, is non-decreasing with $n$, the optimal replacement policy for the family M is to replace a car upon failure or after some replacement period $s^n_M$ which ever comes first. The value $s^n_M$ is determined by the values of $C^n_M$, $c^n_M$, $n = 1, 2, \ldots$, the discount factor $\beta$, and the life distribution function $F^n_M$, $n = 1, 2, \ldots$. The same is true for the family F with the corresponding values of $C^n_F$, $c^n_F$, and $F^n_F$, $n = 1, 2, \ldots$. The above statement about optimal policy is well-known and can be found in books on operation research or Markov decision processes (see Jorgenson et al (1967), Dynkin and Yushkevich (1979), Heyman and Sobel (1982)). Assume that the cars of both companies are identical in terms of their cost characteristics, i.e.

$$C^n_M = C^n_F, c^n_M = c^n_F, n = 1, 2, \ldots \quad (1)$$
but the cars of company M are more reliable than the cars of company F, i.e.

$$F^M_n \leq F^F_n, \ n = 1, 2, \ldots$$  \hspace{1cm} (2)

where \(F^M_n\), \((F^F_n)\) is the probability of failure not later than the \(n\)th period of use for car M, (F).

The aim of this paper is to show that under assumptions (1) and (2), independent of the form of the distribution functions \(F^M\), \(F^F\), the optimal replacement period for the more reliable cars is always less than that for the less reliable cars, i.e.

$$s^M \leq s^F.$$  \hspace{1cm} (3)

Since the actual (random) time of use of a car of each type is equal to or less than the optimal replacement period, \(\tau^M \leq s^M\), \(\tau^F \leq s^F\), inequality (3) does not imply that the expected period of work \(E\tau^M\) for the more reliable car M is less than the expected period \(E\tau^F\) for the less reliable car F.

We may anticipate that as a rule, for most distributions and cost variables, the "normal" situation holds,

$$E\tau^M \geq E\tau^F.$$  \hspace{1cm} (4)

Thus the title of the paper is exaggerated and in general applicable only to the optimal replacement periods. But it is possible, and we present a simple example, when not only the replacement period, but even the expected period of use of a more reliable car is less than the expected period for the less reliable one. It means that if both families follow optimal strategies, family M will consume more copies of reliable cars M than the family F will consume unreliable cars F. But of course M will pay in total less than F. So in some specific cases the title of the paper is justified even with the interpretation of "the period of its work" as the expected period of work. Of course we do not apply the title to a particular copy of a reliable machine.

The outline of our paper is as follows. In Section 2 we describe the model of optimal replacement and the structure of the optimal policy. In Section 3 we prove the main result of the paper, inequality (3), which is called Theorem 1. In Section 4 we present an example when inequality (4) fails to hold. Section 5 contains some concluding remarks and open problems. The statements of Section 2 are well-known, but we present unified and shorter proofs. The proof of Theorem 1 is very simple, but we have failed to find a statement equivalent to (3) in the vast literature related to optimal replacement policies. A similar but weaker statement was formulated by the author in Sonin and Aliev (1984).
2 A simple stochastic model of optimal replacement

A machine operates with fixed productivity in all periods of time and is subjected to failure. If the machine fails it is to be replaced immediately at cost $C$ by a new one. If the machine did not fail then at the end of $i$th period, $i = 1, 2, \ldots$ there are two options: to replace the machine by a new one at a cost $C$, or to proceed with the current machine for the $(i+1)$th period of time at operating cost $c_i$. The case of a deteriorating machine can be included partially in this model by changing the operating costs. We assume that operating costs are non-decreasing in time, i.e.

$$0 \leq c_1 \leq c_2 \leq \ldots$$ (5)

Our convention is that failure and replacement take place at the end of the time period. Denote by $F_n$, $n = 1, 2, \ldots$ the distribution function of failure times of a machine and by $q_n = 1 - F_n$, $n = 1, 2, \ldots$ the survival probabilities. So $1 \geq q_1 \geq q_2 \geq \ldots$ In the deterministic case we have $q_1 = q_2 = \ldots = 1$. The replacement policy that minimizes the total expected discounted cost over an infinite time interval is called optimal. The policy requiring the replacement of each machine upon failure or after $t$ periods of use, whichever comes first, is denoted by $\pi_t$ and $t$ is referred to as the replacement period.

The general theory of Markov decision processes can be used to justify the intuitively clear fact that the optimal policy is a $\pi_t$ policy for some $t$ (see Dynkin and Yushkevich (1979)). Let $\tau_1, \tau_1 + \tau_2, \ldots, \tau_1 + \tau_2 + \ldots + \tau_n$ be the replacement times for a policy $\pi_t$. Then the (random) discounted cost $C_t$ is

$$C_t = (C + \beta c_1 + \ldots + \beta^{\tau-1}c_{\tau-1}) + \beta^{\tau_1}(C + \beta c_1 + \ldots + \beta^{\tau_2-1}c_{\tau_2-1}) + \ldots$$ (6)

Using the strong Markov property of the process of replacement we can rewrite (6) as

$$C_t = (C + \beta c_1 + \ldots + \beta^{\tau-1}c_{\tau-1}) + \beta^{\tau}C'_t,$$ (7)

where $\tau = \tau_1$ is the Markov moment of first replacement, $C_t, C'_t$ are identically distributed random variables and $C'_t$ represents the total discounted cost after time $\tau$, and is independent of the $\sigma$-algebra $F_\tau$, (the $\sigma$-algebra of observations up to and including time $\tau$).

Taking the expectations of both parts of (7), using the independence of $\tau$ and $C'_t$ and the equality $EC_t = EC'_t$ we get

$$D_t \equiv EC_t = E \sum_{i=0}^{\tau} \beta^{i-1}c_{i-1} + (EC_t)(E\beta^{\tau}).$$ (8)
It is easy to see that for the policy \( \pi_t \), the time of first replacement \( \tau \) has the following distribution:

\[
P(\tau = i) = q_{i-1} - q_i, i = 1, ..., t - 1, P(\tau = t) = q_{t-1}.
\]

and hence using standard probabilistic identities, we have

\[
E \sum_{i=0}^{\tau} \beta^{i-1} c_{i-1} = \sum_{i=0}^{t} \beta^{i-1} P(\tau \geq i) = \sum_{i=0}^{t} \beta^{i-1} c_{i-1} q_{i-1},
\]

(9)

\[
1 - E \beta^\tau = (1 - \beta) E \sum_{i=0}^{\tau} \beta^{i-1} = (1 - \beta) \sum_{i=0}^{t} \beta^{i-1} q_{i-1}.
\]

(10)

Using (9), (10) we can rewrite (8) as

\[
D_t = \frac{1}{1 - \beta} C + \beta q_1 c_1 + ... + \beta^{t-1} q_{t-1} c_{t-1} = \frac{1}{1 - \beta} L_t,
\]

(11)

where \( L_t \) is the expected average discounted cost. The optimal replacement period and corresponding optimal replacement policy are specified by the value \( t \) minimizing \( D_t \), or equivalently, \( L_t \), over \( t = 1, 2, ... \) By considering \( L_t \) we allow ourselves an opportunity to treat in the same way the case of the average cost, corresponding to the case \( \beta = 1 \). So, from now on we assume \( 0 < \beta \leq 1 \). The value of the minima is described by Lemma 1.

**Lemma 1.** Let \( s = \min\{t : L_t \leq c_t\} \). Then \( L_t \) is a decreasing sequence for \( t = 1, 2, ..., s \) and increasing for \( t = s, s+1, ..., \) and hence \( s \) is the optimal replacement period for the stochastic case.

**Proof.** We need two simple technical propositions. The first one is the well-known properties of arithmetic proportions.

**Proposition 1.** Let \( b > 0, d > 0 \). Then (a) \( c/d < a/b \) implies \( c/d < (a+c)/(b+d) < a/b \); (b) \( c/d < (a+c)/(b+d) \) implies \( (a+c)/(b+d) < a/b \).

**Proposition 2.** Let \((c_n), (m_n)\) be two sequences of real numbers, \( m_n \) positive, \( n = 0, 1, 2, ... \). Denote by

\[
M_t = \frac{m_0 c_0 + m_1 c_1 + ... + m_{t-1} c_{t-1}}{m_0 + m_1 + ... + m_{t-1}}, t = 1, 2, ...
\]

(12)

Then (a) \( c_t \leq M_t \) implies \( c_t \leq M_{t+1} \leq M_t \), and (b) \( M_t \leq c_t \) implies \( M_t \leq M_{t+1} \leq c_t \).

**Remark 1.** Proposition 2 is obvious if the values of \( m_n, n = 0, 1, 2, ... \) are interpreted as masses of bodies and \( c_n, n = 0, 1, 2, ... \) as their locations on a horizontal axis. Then \( M_t \) is the coordinate of the center of gravity for the first \( t \) bodies, \( n = 0, 1, ..., t - 1 \). The statements (a) and (b) describe the shift of position of the center of gravity if the \((t+1)\)th body is added. The formal proof of Proposition 2 is as follows.
(a) Take $M_t = a/b, c_t = c_t m_t/m_t = c/d$ and apply Part (a) of Proposition 1.
(b) Take $M_t = -a/b, c_t = c_t m_t/m_t = -c/d$ and apply Part (a) of Proposition 1.

Let $C = c_0, \beta q_t = m_t$ for $i = 0, 1, ..., \text{Then } L_t$ in (11) can be represented as $M_t$ in (12). Applying Part (a) of Proposition 2 for $t = 1, 2, ..., s - 1$ and Part (b) for $t = s, s + 1, ...$ we obtain Lemma 1.

Remark 2. It is easy to see from the proof that the monotonicity of $c_n$ for $n = 1, 2, ..., s$ is not used at all and instead of $c_s \leq c_{s+1} \leq ...$ it is sufficient to assume that $c_s \leq c_t$ or even that $L_s \leq c_s$ for $t = s, s + 1, ...$

3 Comparisons of two machines

Let us assume that we have two types of machines with identical cost characteristics, $C^{(1)} = C^{(2)}, c^{(1)}_n = c^{(2)}_n, n = 1, 2, ..., \text{and with possible different distribution functions, } F^{(1)}_n, F^{(2)}_n, n = 1, 2, ...$ for failure times. (Further, $f^{(k)} = f^k$ for any $f$). Let $s^1$ and $s^2$ be the optimal replacement periods for these machines. We recall once more that a machine of each type can be replaced only by a machine of the same type.

Theorem 1. Suppose that machines of the first type are more reliable, i.e. $F^{(1)}_n \leq F^{(2)}_n$ for all $n = 1, 2, ...$ Then

$$s^1 \leq s^2.$$  \hspace{1cm} (13)

To prove Theorem 1 we need only Proposition 3.

Proposition 3. Let $(c_n), (m_n), (u_n), n = 0, 1, ..., t - 1$ be three sequences of real numbers, with $(m_n), (u_n)$ positive. Let $M^m_t, M^u_t$ be defined by formula (12) for $(m_n)$ and $(u_n)$. Suppose for each $k = 0; 1; ...; t - 1$ the following conditions hold:

$$u_k \leq m_k \text{ if } c_k \leq M^m_t \text{ and } m_k \leq u_k \text{ if } c_k \geq M^m_t \hspace{1cm} (14)$$

Then

$$M^m_t \leq M^u_t.$$ \hspace{1cm} (15)

Remark 3. (See Remark 1.) Proposition 3 is rather evident if the values of $c_n, m_n, n = 0, 1, ..., t - 1$ are interpreted as two systems of masses with the same coordinates $c_n, n = 0, 1, ..., t - 1$ and $M^m_t, M^u_t$ as the centers of gravity for two subsystems consisting of the $t$ bodies. We can obtain the second subsystem by decreasing all the masses lying to the left of the center of gravity of the first subsystem $M^m_t$ and increasing all the masses lying to the right of that point. This shifts the center of gravity to the right. The formal
proof of the lemma can be easily carried out using Proposition 1. **Proof of Theorem 1.** By Lemma 1 the optimal replacement periods \( s^1, s^2 \) are given by \( s^k = \min(t : L^k_t \leq c_t) \), where \( L^k_t, k = 1, 2 \) are the expected discounted average costs (see (11)). Hence to show (13), it suffices to prove that \( L^1_t < L^2_t \) for \( t = 2, ..., s^1 - 1, (L^1_1 = L^2_1 = C) \). Define the sequences \((m_n), (u_n)\) by the equalities \( m_n = \beta^n q^1_n, \ u_n = \beta^n q^2_n \) where \( q^k_n \) is the survival function for the machine of type \( k, k = 1, 2 \). Let \( c_0 = C \) and represent the functions \( L^1_t, L^2_t \) in (11) as the functions \( M^m_t, M^u_t \) in (12). By assumption \( F^1_n \leq F^2_n, n = 1, 2, ... \) and hence \( q^1_n = 1 - F^1_n \geq q^2_n = 1 - F^2_n \). Therefore \( u_n \leq m_n, n = 1, 2, ... \) and since \( c_t < M^m_t \equiv L^1_t, t = 1, 2, ..., s^1 - 1 \) and \( m_0 = u_0 \) we have that condition (14) holds at all points \( t = 1, 2, ..., s^1 - 1 \). Applying Proposition 3 we obtain the inequalities \( L^2_t \equiv M^t_t > M^m_t \equiv L^1_t, t = 1, 2, ..., s^1 - 1 \) and therefore the theorem.

**Remark 4.** It can be shown that inequality \( L^1_t < L^2_t \) holds not only for \( t < s^1 \) but for \( s^1 \leq t \leq s^2 \) also. In general it fails to hold for \( t > s^2 \).

4 An example of when inequality (4) does not hold.

Let \( \beta = 0.9, C = 3, c_1 = 1, c_2 = 2, c_3 = 2.2 \leq c_i, i = 4, ... \) and suppose the first (reliable) machine is not subjected to failure for the first two periods, i.e. \( q^1_1 = q^1_2 = 1 \). Also suppose that \( q^2_1 = q^2_2 = 0.8 \), i.e. the second (unreliable) machine is subjected to failure with probability 0.2 at the first period. Subsequent values of \( q_i, i \geq 3 \) are irrelevant. By (11) for the first machine we have \( L_1 = 3, L_2 = 3.9/1.9 \approx 2.05 < c_2 = 2.1 \) and hence the optimal replacement period for the first machine is \( s^{(1)} = 2 \) and coincides with the expected period of work of each copy of a machine of the first type. For the second machine we have \( L_1 = 3, L_2 = 3.72/1.72 \approx 2.16 > c_2 = 2.1, L_3 = 5.0808/2.368 \approx 2.15 < c_3 = 2.2 \). Hence the optimal replacement period for the second machine is \( s^{(2)} = 3 \) and the expected period of use for each copy of a machine of type 2 is \( E\tau = 1(0.2) + 3(0.8) = 2.6 \), which exceeds the expected period of work for the first (reliable) machine. It is also easy to check that if all parameters remain unchanged except \( \beta \), and \( \beta = 0.9 \) is replaced by \( \beta = 1 \), we will get the same values \( s^{(1)} = 2 \) and \( s^{(2)} = 3 \) and hence the same effect holds for the case of minimizing average cost.
5 Concluding remarks and some open problems

Despite the paradoxical form of the title, the main result of the paper is in full accordance with economic considerations and common sense. The rapid replacement of equipment is economically profitable only if operating costs for some initial period of time are relatively small and the new equipment is reliable. The last point is to ensure the absence of losses due to possible failure. If a machine is not reliable (not only a particular copy, but a type) then it is more profitable to continue with a working machine, even with relatively higher operating cost than to replace it with an unreliable one. As the example in Section 4 shows, it is possible to be in the actually paradoxical situation when for the same amount of demand of work, an industry has to produce more copies of a reliable machine than an unreliable one. (By 30% in this example). It is not clear whether such a situation is possible for similar models in continuous time. It seems that the general techniques elaborated in Anderson (1988) for continuous time may be appropriate for treating these questions. The more general problem is to find and describe the classes of distributions and operating costs for which the expected life length of a more reliable machine actually exceeds the expected life length for a less reliable one. Another possible direction of study is the comparison of two types of machines when there is a possibility to replace acting machines by machines with improved characteristics and in a more general setting as considered, for example in Hopp and Nair (1991) and Bylka et al. (1992).

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References


