You must work three of the following four questions for full credit.

1. Do the following
   a) Let $\mu$ be a finite measure on Borrel $\sigma$-algebra of subsets of $\mathbb{R}$. The set $S(\mu)$ is called the support of $\mu$ if for any $x \in S(\mu)$ and arbitrary $\epsilon > 0$, $\mu([x-\epsilon,x+\epsilon]) > 0$. Prove that $S(\mu)$ is a closed set.
   b) Let $\mu$ consist of the atoms at the points $\{x_n = \frac{1}{2^n}, n = 0, 1, 2, \cdots\}$ with $\mu(\{x_n\}) = \frac{1}{2^n}$. Describe the support of $\mu$. Find the measure $\mu(S(\mu))$. 

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2. Let $f(x)$ be a Lebesgue measurable function on $[0, 1]$.
   a) Prove that $f^3(x)$ is also Lebesgue measurable on $[0, 1]$.
   b) Provide an example of $g$ such that $g^2(x)$ is Lebesgue measurable, but $g(x)$
       is not Lebesgue measurable on $[0, 1]$. 
3. Let \( \{f_n\} \) and \( \{g_n\}, g_n \geq 0, \) be sequences of Lebesgue \( (\mu) \) measurable functions on \([0,1]\) that converge pointwise to \( f \) and \( g \) respectively. Suppose further that

\begin{align*}
&\text{a} \quad |f_n| \leq g_n, \forall n \in \mathbb{N} \quad \text{and} \\
&\text{b} \quad \lim_{n \to \infty} \int_{[0,1]} g_n d\mu = \int_{[0,1]} g d\mu < \infty.
\end{align*}

Prove that

\[
\lim_{n \to \infty} \int_{[0,1]} f_n d\mu = \int_{[0,1]} f d\mu.
\]
4. Let \( f(x) \) be a monotonically increasing function on \([0,1]\). Let \( S \) be the set of points of discontinuity of \( f(x) \). For \( c \in (0,1) \) we write

\[
d(c) \equiv \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x).
\]

a) Let \( 0 < c_1 < c_2 < \cdots < c_k < 1 \) be \( k \) points in \( S \). Prove that

\[
d(c_1) + \cdots + d(c_k) \leq f(1) - f(0).
\]

b) Prove that \( S \) is a countable set.