You must turn in exactly three of the following four problems. Clearly identify the three problems that you want graded. The solution of each problem should be submitted on a separate page (or pages) with your name and the problem number on each page.

(Example: For problem 3 on the Real Analysis I portion, label the solution page(s) “RAI.3”)

(Example: For problem 3 on the Real Analysis II portion, label the solution page(s) “RAII.3”)

II.1) Let $X$ be a compact space and let $A$ be a closed subset of $X$. Prove directly from the definition that $A$ is compact.

II.2). Let $H$ be a Hilbert space with inner product $(x,y)$ for $x,y \in H$ and let $\{x_n\} \subset H$ such that $\{x_n\}$ converges weakly to $x \in H$. Prove that $\|x_n - x\| \to 0$ if and only if $\lim_{n \to \infty} \|x_n\| = \|x\|$. 

II.3). Let $H$ be a Hilbert space with inner product $(x,y)$ for $x,y \in H$ and let $\{x_n\} \subset H$ such that $\{x_n\}$ converges weakly to $x \in H$. Prove that $\{\|x_n\|\}$ is bounded, i.e. there is a constant $M$ independent of $n$ such that $\|x_n\| \leq M$.

II.4). Let $(X,B,\mu)$ be a complete measure space and let $f \in L^p(\mu)$, $1 < p < \infty$. Let $\{T_n\}$ be a convergent sequence of bounded operators on $L^p(\mu)$, i.e. there exists a $T \in \mathcal{L}(L^p(\mu))$ such that $\|T_n - T\| \to 0$. Let $g \in L^q(\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$, and suppose there exists $\{g_n\} \subset L^q(\mu)$ such that $\|g_n - g\|_q \to 0$. Prove that $\int_X (T_nf)g_n \, d\mu \to \int_X (Tf)g \, d\mu$. 

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