You must work on three and only three of the following four questions for full credit. If you work on all four questions, only the first three will be graded.
1. [10 points] Let $f$ be a Lebesgue ($\mu$) integrable function on $\mathbb{R}$ and $n \in \mathbb{N}$.

(a) Let $E_n = \{ t \in \mathbb{R} : f(t) \geq n \}$. Prove that $\lim_{n \to +\infty} \int_{E_n} f \, d\mu = 0$.

(b) Prove that $\lim_{n \to +\infty} n \cdot \mu(E_n) = 0$. 
2. [10 points] Let $\mu$ be the Lebesgue measure on $\mathbb{R}$ and $f : E \to \mathbb{R}$ be a function with a measurable domain $E$. Consider the following three statements.

**Statement (a)** $f^{-1}((\alpha, \infty))$ is measurable for each $\alpha \in \mathbb{R}$;

**Statement (b)** $f^{-1}([\alpha, \infty))$ is measurable for each $\alpha \in \mathbb{R}$;

**Statement (c)** $f^{-1}(\{\alpha\})$ is measurable for each $\alpha \in \mathbb{R}$.

(1) Prove that Statements (a) and (b) are equivalent.

(2) Does Statement (c) imply that $f$ is a measurable function? Provide your proof (if your answer is 'yes') or a counterexample (if your answer is 'not always true').
3. [10 points] Let $f(x)$ be a Lebesgue measurable function on $[0,1]$. For $n \in \mathbb{N}$, define 
\[ \varphi_n(x) = \tan^{-1}(nf(x)), x \in [0,1]. \]

(a) Prove that the limit function $\varphi(x) = \lim_{n \to \infty} \varphi_n(x)$ is well defined, and that 
\[ \int_{[0,1]} \varphi d\mu = \frac{\pi}{2} \left( \mu([0,1] : f(x) > 0) - \mu([0,1] : f(x) < 0) \right). \]

(b) Prove that 
\[ \lim_{n \to +\infty} \int_{[0,1]} \varphi_n d\mu = \int_{[0,1]} \varphi d\mu. \]
4. [10 points] Do the following.

(a) State the definition of an absolutely continuous function on a finite interval \([a, b]\).

(b) Let \(f(t)\) be a Lebesgue integrable function on \(\mathbb{R}\). Assume that for all rational numbers \(\alpha, \beta\) with \(\alpha < \beta\), \(\int_{\alpha}^{\beta} f(t)dt = 0\). Prove that \(f(t) = 0\) almost everywhere on \(\mathbb{R}\).