Real Analysis I Qualifying Examination

(August 19, 2015)

UNCC ID: ______________________

You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to clearly mark the question that you don’t want to be graded. Otherwise, only the first three will be graded.
1. [10 points] Let $E$ be a subset of $\mathbb{R}$ with finite outer measure $m^*(E)$ and $a < b < c$.

   (a) Prove that
   
   $$m^*(E \cap [a, c]) = m^*(E \cap [a, b]) + m^*(E \cap [b, c]).$$

   (b) Define the function $f_E$ by
   
   $$f_E(x) \equiv m^*(E \cap (-\infty, x]), \quad \forall x \in \mathbb{R}.$$ 

   Prove that $f_E$ is increasing and continuous.
2. [10 points] Let $f(x)$ and $g(x)$ be Lebesgue measurable functions on $[0, 1]$.

(a) Prove by definition that $|f(x)|$ is Lebesgue measurable on $[0, 1]$.

(b) Prove by definition that $h(x) \equiv \min(f(x), g(x))$ is Lebesgue measurable on $[0, 1]$. 
3. [10 points] Let \( f_n(x) \equiv \frac{\sin x}{n^{-2} + \cos x} \), \( x \in \left[0, \frac{\pi}{2}\right] \), \( n \in \mathbb{N} \), and let \( \nu(x) \) be a bounded non-negative measurable function on \( \mathbb{R} \). Do the following.

(a) Prove that the limit \( \lim_{n \to \infty} \int_{\pi/4}^{\pi/3} f_n(x) \, dx \) exists.

(b) Find the limit value in (a).

(c) Prove that, for \( \alpha \in (0, \pi/2) \),

\[
\lim_{n \to \infty} \int_0^\alpha f_n(x) \nu(x) \, dx = \lim_{n \to \infty} \int_0^\alpha \tan(x) \nu(x) \, dx,
\]

and this limit is finite.
4. [10 points] Do the following.

   (a) State the definition of an absolutely continuous function on a finite interval \([a, b]\).

   (b) Let \(P(x)\) be a polynomial function on \([0, 1]\). Prove that \(P(x)\) is absolutely continuous on \([0, 1]\).