You must work on three and only three of the following four questions for full credit. If you work on all four questions, only the first three will be graded.
1. [10 points] Do the following.

(a) Let $E \subset \mathbb{R}$. State the definition of the outer measure $m^*(E)$ of $E$.

(b) Let $E \subset \mathbb{R}$ and $0 < m^*(E) < \infty$. Prove that there exists an open interval $I = (a, b)$ with the property that

$$m^*(E \cap I) > 0.99 \ m^*(I).$$
2. [10 points] Let \( \{f_n, n \in \mathbb{N}\} \) be Lebesgue measurable functions on \([0, 1]\).

(a) Prove that the function \( f(x) \equiv \sup\{f_n(x), n \in \mathbb{N}\} \) is a measurable function.

(b) Let \( E \subset [0, 1] \) be the set of points where the sequence \( \{f_n(x), n \in \mathbb{N}\} \) converges. Prove that \( E \) is Lebesgue measurable.
3. [10 points] Define

\[ f_n(x) \equiv \frac{3 + \exp(n \sin x)}{2 + \exp(n \sin x)}, \quad x \in [0, 2\pi]. \]

(a) Prove that

\[ \lim_{n \to +\infty} \int_{[0, 2\pi]} f_n = \int_{[0, 2\pi]} \lim_{n \to +\infty} f_n. \]

(b) Find the value of the above limit.
4. [10 points] Do the following.

(a) State the definition of a function of bounded variation on \([a, b]\).

(b) Let \(f\) be the function defined by

\[
    f(x) = \begin{cases} 
        0 & \text{if } x = 0; \\
        x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0.
    \end{cases}
\]

Prove that this function is not of bounded variation on the interval \([0, \frac{2}{\pi}]\).