You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to mark clearly the question that you don’t want to be graded. Otherwise, only the first three will be graded.

Unless otherwise stated, you may appeal to a well-known theorem by name in your solution to a problem. However, if you do so, then it is your responsibility to make it clear which theorem you are using and why its use is justified.
1. Suppose that $E$ is a Lebesgue measurable subset of $\mathbb{R}$ with finite measure, and let $1 \leq p_1 < p_2 < \infty$.

(a) Use Hölder’s Inequality to show that $L^{p_2}(E) \subseteq L^{p_1}(E)$.

(b) Show that if $(f_n)_{n \geq 1} \to f$ in $L^{p_2}(E)$, then $(f_n)_{n \geq 1} \to f$ in $L^{p_1}(E)$. 
2. Suppose that $X$ is a Banach space and $P : X \to X$ is a bounded linear map such that $P \circ P = P$ (where $\circ$ denotes composition). Let $Y = P(X)$. Prove that $P : X \to Y$ is an open mapping.
3. Suppose that $H$ is a Hilbert space and $V$ is a nonempty closed subset of $H$ such that if $u, v \in V$ then $\lambda u + (1 - \lambda)v \in V$ for all $\lambda \in [0, 1]$. Prove that there exists a unique $u \in V$ such that $\|u\| \leq \|v\|$ for all $v \in V$. You may use the Parallelogram Identity:

$$\|a - b\|^2 + \|a + b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$
4. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{L}(H)$ be the set of bounded linear operators from $H$ to itself.

(a) Let $T \in \mathcal{L}(H)$, and suppose that $(T_n)_{n \geq 1} \subset \mathcal{L}(H)$ is a sequence of compact operators such that $\|T_n - T\|$ converges to zero. Prove that $T$ is compact.

(b) Suppose that $(\varphi_k)_{k \geq 1}$ is an orthonormal basis of $H$, and let $(\lambda_k)_{k \geq 1}$ be a sequence of real numbers that converges to zero. Define $T : H \to H$ by

$$T(h) = \sum_k \lambda_k \langle h, \varphi_k \rangle \varphi_k.$$

Prove that $T$ is compact.