You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to clearly mark the question that you don’t want to be graded. Otherwise, only the first three will be graded.
1. [10 points] Let $f$ be a non negative Lebesgue measurable function on $E$ with $m(E) < \infty$. Denote $E_k = f^{-1}([k, k + 1)), k \in \mathbb{N}$. Prove that

$$\int_E f < +\infty$$

if and only if

$$\sum_{k=1}^{\infty} k \cdot m(E_k) < +\infty.$$
2. [10 points] Do the following.

(a) Give a definition of a Lebesgue measurable function on \( \mathbb{R} \).

(b) Assume that the function \( f \) defined on \((0, 1)\) is differentiable at each point on \((0, 1)\). Prove that its derivative \( f' \) is Lebesgue measurable on \((0, 1)\).

(c) Let \( f \) be any function defined on \( \mathbb{R} \). Define a new function \( g \) by

\[
g(x) \equiv \chi_{\mathbb{Q}}(x) \cdot f(x), \quad x \in [0, 1],
\]

where \( \mathbb{Q} \) is the set of rational numbers and \( \chi_{\mathbb{Q}} \) is the characteristic function of \( \mathbb{Q} \). Prove that \( g \) is Lebesgue measurable on \( \mathbb{R} \).
3. [10 points] Let $m$ and $n$ be natural numbers $m, n \geq 1$ and $m < n$. Assume that $E_1, E_2, \cdots, E_n$ are $n$ Lebesgue measurable subsets of $[0, 1]$ with the property that each point $x \in [0, 1]$ is in at least $m$ of above subsets. Prove that there exists at least one $E_j$ such that $m(E_j) \geq \frac{m}{n}$.
4. [10 points] Do the following.

(a) State the definition of bounded variation.

(b) Let $f$ be the function defined by

$$f(x) = \begin{cases} 
  0, & \text{if } x = 0; \\
  1 - x, & \text{if } x \in (0, 1); \\
  7, & \text{if } x = 1. 
\end{cases}$$

Prove that $f$ is of bounded variation. Find the total variation of $f$.

(c) Write the function $f$ into the form of difference of two increasing functions.