You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to mark clearly the question that you don’t want to be graded. Otherwise, only the first three will be graded.

Unless otherwise stated, you may appeal to a well-known theorem by name in your solution to a problem. However, if you do so, then it is your responsibility to make it clear which theorem you are using and why its use is justified.
1. Suppose that $E$ is a Lebesgue measurable subset of $\mathbb{R}$, $1 \leq p < \infty$, and $q$ is the conjugate of $p$. Further, suppose that

- $\{f_n\}_{n \geq 1}$ converges weakly to $f$ in $L^p(E)$ (i.e. for each bounded linear functional $\psi : L^p(E) \to \mathbb{R}$, we have $\psi(f_n) \to \psi(f)$), and
- $\{g_n\}_{n \geq 1}$ converges strongly to $g$ in $L^q(E)$.

Prove that

$$\lim_{n \to \infty} \int_E g_n \cdot f_n = \int_E g \cdot f.$$
2. Let $X$ be a normed linear space, let $Y \subseteq X$ be a closed linear subspace, and let $x_0 \in X \setminus Y$. Prove that there exists a bounded linear functional $\psi : X \to \mathbb{R}$ such that $\psi(Y) = 0$ and $\psi(x_0) \neq 0$. 
3. Let $X$ be a Banach space, and suppose that $\{x_n\}_{n \geq 1}$ is a sequence in $X$ such that $\sum_n \|x_n\| < \infty$.

(a) Prove that the series $\sum_n x_n$ converges in $X$.

(b) Prove that if $T : X \to X$ is a bounded linear operator, then $T(\sum_n x_n) = \sum_n T(x_n)$. 
4. Let $H$ be a Hilbert space, and let $K : H \to H$ be a compact linear operator from $H$ to itself.

(a) Prove that if $\{h_n\}_{n \geq 1}$ is bounded, then $\{K(h_n)\}_{n \geq 1}$ has a convergent subsequence.

(b) Prove that if $\{h_n\}_{n \geq 1}$ is weakly convergent, then $\{K(h_n)\}_{n \geq 1}$ is strongly convergent.