You must work three of the following four questions for full credit. Before handing in your exam, please mark which one you don’t want to be graded.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $G \subset \mathbb{R}$ is open, and $F \subset \mathbb{R}$ is closed.
   a) Prove from the definition that $f^{-1}(G)$ is open.
   b) Prove from the definition that $f^{-1}(F)$ is closed.
   c) Give an example of a bounded open set $G$ and a continuous function $g : G \to \mathbb{R}$ such that $g(G)$ is closed.
2. Let $E$ be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) > 0$, and let $c \in (0, 1)$. Prove that there exists a nonempty open interval $(a, b)$ with the property that 

$$m(E \cap (a, b)) \geq c \cdot m((a, b)).$$
3. a) State the Lebesgue Dominated Convergence Theorem (LDCT).

b) Let
\[ f_n(x) \equiv \frac{1}{n} \cdot \frac{1}{\frac{1}{n^2} + x^2} = \frac{n}{1 + (nx)^2}, \quad x \in [0, \infty), \quad n \in \mathbb{N}. \]

Prove that
\[ \int_{[0, \infty)} \lim_{n \to +\infty} f_n \neq \lim_{n \to +\infty} \int_{(0, \infty)} f_n. \]

c) Explain why in the example b) one can not use the LDCT as you stated in a).
4. a) Let \( f : [0, 1] \rightarrow \mathbb{R} \) be an increasing, continuous function such that \( f \) is absolutely continuous on the interval \( \left[ \frac{1}{n}, 1 \right] \) for each \( n \in \mathbb{N} \). Prove that \( f \) is absolutely continuous on \( [0, 1] \).

b) Give an example of a continuous function \( f : [0, 1] \rightarrow \mathbb{R} \) such that \( f \) is absolutely continuous on the interval \( \left[ \frac{1}{n}, 1 \right] \) for each \( n \in \mathbb{N} \), and yet \( f \) is not absolutely continuous on \( [0, 1] \). (You should prove that your example has these properties.)