You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to mark clearly the question that you don’t want to be graded. Otherwise, only the first three will be graded.

Unless otherwise stated, you may appeal to a well-known theorem by name in your solution to a problem. However, if you do so, then it is your responsibility to make it clear which theorem you are using and why its use is justified.
1. Let $m^*$ denote the Lebesgue outer measure, and suppose that $E \subset \mathbb{R}$ satisfies $m^*(E) < \infty$.

(a) State the definition of $m^*(E)$.

(b) Prove that for each $\epsilon > 0$, there exists $M > 0$ such that $m^*(E \setminus [-M, M]) < \epsilon$.

(c) Prove that $E$ is measurable if and only if for each $\epsilon > 0$, there exists a compact set $K \subset E$ such that $m^*(E \setminus K) < \epsilon$. 


2. Suppose that $g : E \to \mathbb{R}$ is measurable.
   (a) Prove that for all $t > 0$,
   \[
   m\left(\{x \in E : |g(x)| > t\}\right) \leq \frac{1}{t} \int_E |g|
   \]
   (b) Now suppose that $\int_E |g| = 0$ and prove that $g = 0$ a.e. on $E$. 

3. (a) Suppose that \( \{f_n\}_n \) is a sequence of non-negative measurable functions on \([0, 1]\) and \( f_n \to f \) pointwise a.e. on \([0, 1]\), where \( f \) is integrable on \([0, 1]\). Let \( g_n(x) = \min\{f_n(x), f(x)\} \) for all \( n \in \mathbb{N} \) and \( x \in [0, 1] \). Prove that each \( g_n \) is integrable and find
\[
\lim_n \int_0^1 g_n.
\]

(b) Give an example of a sequence \( \{f_n\}_n \) of non-negative measurable functions on \([0, 1]\) such that \( f_n \to 0 \) pointwise a.e. on \([0, 1]\) and yet \( \int_0^1 f_n \) does not converge to 0. Remember to prove that your example has these properties.
4. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be absolutely continuous.

(a) Show that the product $f \cdot g$ is absolutely continuous.

(b) Show that the following integration by parts formula holds:

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$