You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to mark clearly the question that you don’t want to be graded. Otherwise, only the first three will be graded.

Unless otherwise stated, you may appeal to a well-known theorem by name in your solution to a problem. However, if you do so, then it is your responsibility to make it clear which theorem you are using and why its use is justified.
1. Suppose that $E$ is a Lebesgue measurable subset of $\mathbb{R}$ with $0 < m(E) < \infty$ and $f \in L^p(E)$ for all $1 \leq p \leq \infty$. Prove that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$
2. Suppose $X$ and $Y$ are Banach spaces. Let $\{\ell_n\}_n$ be a sequence of bounded linear functionals on $X$, and let $\{y_n\}_n$ be a sequence in $Y$ such that for each $x \in X$, the following series converges in $Y$:

$$
\sum_{n=1}^{\infty} \ell_n(x)y_n.
$$

Let $S : X \to Y$ be the map defined by

$$
S(x) = \sum_{n=1}^{\infty} \ell_n(x)y_n.
$$

Prove that $S$ is a bounded linear operator.
3. Let $X$ be a Banach space. Let $\{y_j\}_j$ be a subset of $X$, and let $Y = \text{span}\{y_j\}$ (the closed linear span of $\{y_j\}_j$). Let $x_0 \in X$. Prove that the following statements are equivalent:

(i) $x_0$ is in $Y$

(ii) for every bounded linear functional $\ell : X \to \mathbb{R}$, if $\ell(y_j) = 0$ for all $j$, then $\ell(x_0) = 0$. 
4. Let \( \{ \varphi_k \}_k \) be an orthonormal basis of the Hilbert space \( H \), and let \( \{ u_n \}_n \) be a bounded sequence in \( H \). Prove that \( \{ u_n \}_n \) converges to 0 weakly in \( H \) if and only if for each \( k \),

\[
\lim_n < u_n, \varphi_k > = 0.
\]