You will receive credit for three and only three of the following four questions. If you work on all four questions, please make sure to mark clearly the question that you don’t want to be graded. Otherwise, only the first three will be graded.

Unless otherwise stated, you may appeal to a well-known theorem by name in your solution to a problem. However, if you do so, then it is your responsibility to make it clear which theorem you are using and why its use is justified.
1. Let $A$ and $B$ be subsets of $\mathbb{R}$, and suppose that $A \subset (-\infty, 0)$ and $B \subset (0, \infty)$. Let $m^*$ denote the Lebesgue outer measure. Prove that

$$m^*(A \cup B) = m^*(A) + m^*(B).$$
2. Suppose that $f$ is a measurable function on $E$, $m(E) < \infty$, and $f$ is finite almost everywhere.
   (a) For $n \in \mathbb{N}$, let
   \[ A_n = \{ x \in E : |f(x)| > n \}. \]
   Find $\lim_n m(A_n)$, and justify your answer.

   (b) Prove that for each $\epsilon > 0$, there exists a measurable set $F \subset E$ such that $f$ is bounded on $F$ and $m(E \setminus F) < \epsilon$. 
3. Compute the following limit and justify your answer:

$$\lim_{n \to \infty} \int_1^{\infty} \frac{\sin^n(x)}{x^2} \, dx.$$
4. Suppose that \( \{a_n\}_{n=1}^\infty \) is a sequence of nonnegative numbers, and let \( f : [0, 1] \to \mathbb{R} \) be given by

\[
f(x) = \begin{cases} 
(-1)^n a_n, & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \\
0, & \text{if } x = 0.
\end{cases}
\]

Prove that \( f \) has bounded variation if and only if \( \sum_{n=1}^\infty a_n < \infty \).