You must work on three and only three of the following four questions for full credit. If you work on all four questions, please make sure to mark clearly the question that you don’t want to be graded. Otherwise, only the first three will be graded.

Unless otherwise stated, you may appeal to a well-known theorem by name in your solution to a problem. However, if you do so, then it is your responsibility to make it clear which theorem you are using and why its use is justified.
1. Let $a < b$, and let $p \in [1, \infty)$.
   (a) Prove that there exists $C > 0$ such that for all continuous $f : [a, b] \to \mathbb{R}$,
   $$\|f\|_p \leq C\|f\|_\infty.$$

   (b) Prove that there is no constant $c > 0$ such that for all continuous $f : [a, b] \to \mathbb{R}$,
   $$\|f\|_\infty \leq c\|f\|_p.$$
2. Let \( \{T_n\}_{n=1}^{\infty} \) be a sequence in \( \mathcal{L}(X, Y) \), where \( X \) and \( Y \) are Banach spaces. Prove that the sequence \( \{\|T_n\|\}_{n=1}^{\infty} \) is bounded if and only if for each \( x \in X \), the sequence \( \{T_n(x)\}_{n=1}^{\infty} \) is bounded in \( Y \).
3. Let $H = L^2([0, 1])$, and let $\mathcal{F}$ be an orthonormal subset of $H$.
   (a) Prove that for any $f, g \in \mathcal{F}$,
   $$\|f - g\|_2 = \sqrt{2}.$$  

   (b) Prove that $\mathcal{F}$ must be countable.
4. Let $H$ be a Hilbert space, and let $K : H \to H$ be a linear operator from $H$ to itself. Prove that if $K(H)$ is a finite dimensional subspace, then $K$ is compact.