REAL ANALYSIS PRELIMINARY EXAMINATION FALL 2004
Wednesday, August 18, 2004 9:00 a.m. - 12:00 noon

INSTRUCTIONS: Please choose six (6) out of the following eight (8) problems.
Each solution you submit should be on a separate page (or pages).
Be sure your name and the problem number appear with each solution.

1) Let \( C \) be any collection of disjoint open intervals in \( \mathbb{R} \). Prove that \( C \) is at most
   countable.

2) Construct a set in \( \mathbb{R} \) which is countably infinite, which is neither open nor closed, and
   belongs to both \( G_\delta \) and \( F_\sigma \).

3) (a) Prove that if \( (X, \mathcal{M}, \mu) \) is a measure space, \( 1 \leq p < \infty \), and \( g \in L^p(X, \mathcal{M}, \mu) \) then

   \[ \mu(\{x \in X : |g(x)| \geq t\}) \leq \left( \frac{\|g\|_{L^p}}{t} \right)^p \text{ for all } 0 < t < \infty. \]

   (b) Use the inequality in (a) with \( p = 1 \) to prove that if \( \int_X |g(x)|d\mu = 0 \) then
       \( g(x) = 0 \) a.e.

4) Let \( m \) be Lebesgue measure and \( B \) the Borel sets. Let the measure \( \mu \) on \( ([0, 1], B) \)
   have an absolutely continuous (with respect to \( m \)) component with density \( \rho(x) = x \) and
   a discrete component with atoms at \( 1/4, 1/2, \) and \( 3/4, \) and with respective masses
   \( m_1 = 1/3, m_2 = 1/3, m_3 = 1/3. \) Calculate \( \int_0^1 x^2d\mu. \)

5) Prove that if \( \varphi(x) \) is a positive continuous function on \( \mathbb{R} \) then
   \[ \lim_{n \to \infty} \int_0^\infty \frac{\varphi(x)}{(\varphi(x))^2 + \frac{1}{x^2}} e^{-x}dx = 1. \]

6) Let \( (X, \mathcal{M}, \mu) \) be a \( \sigma \)-finite measure space and let \( L^p = L^p(X, \mathcal{M}, \mu), p \geq 1. \) For a
   set \( A \subset X \) let \( \chi_A \) denote the characteristic function of \( A, \) i.e. \( \chi_A(x) = 1 \) if \( x \in A \) and
   \( \chi_A(x) = 0 \) if \( x \notin A. \)
   (a) Show that if \( f, g \in L^2 \) then \( fg \in L^1. \)
   (b) Suppose that \( f \perp 1_A \forall A \in \mathcal{M} \) with \( \mu(A) < \infty. \) Prove that \( f = 0 \) a.e. (\( \mu \)).

7) Let \( \langle \varphi_n \rangle_{n=1}^\infty \) be an orthonormal system for a Hilbert space \( H. \) Given \( x \in H \) let
   \( \alpha_n = \langle x, \varphi_n \rangle \) where \( ( , ) \) denotes the inner product in \( H. \)
   Prove that \( \lim_{n \to \infty} \alpha_n = 0. \)

8) Let \( X = \{0, 1, 2, 3, 4\} \) and let \( \sum \) be the \( \sigma \)-algebra of all subsets of \( X. \) Define measures
   \( \alpha \) and \( \beta \) on \( \sum \) via the following table:
   \[
   \begin{array}{cccccc}
   & 0 & 1 & 2 & 3 & 4 \\
   \alpha & 1/2 & 0 & 3 & 2 & 0 \\
   \beta & 2/3 & 0 & 8 & 2 & 5 \\
   \end{array}
   \]
   (a) Is \( \alpha \ll \beta? \) Justify your answer.
   (b) Is \( \beta \ll \alpha? \) Justify your answer.

If the answer is yes in either case, find the appropriate Radon-Nikodym derivative(s).