REAL ANALYSIS PRELIMINARY EXAMINATION SPRING 2004
Wednesday, January 7, 2004 9:00 a.m. – 12:00 noon

INSTRUCTIONS: Please choose six (6) out of the following eight (8) problems. Each solution you submit should be on a separate page (or pages). Be sure your name and the problem number appear with each solution.

1. Give an example of sets $A_{i,j}, i = 1,2; j = 1,2$ such that $\bigcup_{i=1}^{2} \bigcap_{j=1}^{2} A_{i,j} \neq \bigcap_{j=1}^{2} \bigcup_{i=1}^{2} A_{i,j}$.

2. Let $A = [0, \frac{1}{4}], B = [\frac{1}{4}, 1], C = [\frac{1}{2}, 1]$.
   a. Determine the algebra $\Gamma$, of sets generated by $\{A, B, C\}$.
   b. Characterize the set of real valued $\Gamma$-measurable functions defined on $[0, 1]$.
   Use Lebesgue measure on $[0, 1]$ for (c) and (d) below:
   c. Determine the expected value of the function $x^2$ with respect to the algebra $\Gamma$, $E(x^2 | \Gamma)$.
   d. Give an example of a nontrivial bounded real valued function $f$ ($f \neq 0$) on $[0, 1]$ such that $E(f | \Gamma) = 0$.

3. Let $X$ be the set of real numbers and let $\mathcal{A}$ be the algebra of finite unions of intervals of the form $(a, b]$. Define $\mu_\theta(a, b] = e^{2b} - e^{a}$ and let $\mathcal{B}$ be the sigma algebra generated by $\mathcal{A}$. Show that $\mu_\theta$ can be extended to a measure $\mu: \mathcal{B} \to \mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure.

4. Let $(X, \mathcal{B})$ be a measurable space, and let $f: X \to X$ be $\mathcal{B}$-measurable; i.e., $\forall S \in \mathcal{B}, f^{-1}(S) \in \mathcal{B}$.
   a. Prove that $f^{-1}(\mathcal{B}) = \{ f^{-1}(S): S \in \mathcal{B} \}$ is a $\sigma$-algebra over $X$.
   b. With $\mathcal{X} = \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ and $\mathcal{B}$ the Borel $\sigma$-algebra over $X$, let $f(x) = |x|$.
      Describe all intervals $[u, v]$ in $f^{-1}(\mathcal{B})$.
   c. (with reference to (b) above) Prove that the function $\cos(x)$ is $f^{-1}(\mathcal{B})$-measurable.

5. Let $H$ be a Hilbert space.
   a. Let $\langle \varphi_n \rangle$ be an orthonormal system in $H$. For each $x \in H$, set $\hat{x}_n = \langle x, \varphi_n \rangle$. Show that $\hat{x}_n \to 0$.
   b. Suppose that $x_n \to x$ in norm in $H$. Show that $x_n \to x$ weakly in $H$ as well.
   c. Produce a counterexample to show that the converse of (b) is not true.
6. Suppose that $E$ is a Lebesgue measurable subset of $R$, $(f_n)_{n \in \mathbb{N}}$ is a sequence of non negative measurable functions, and $f$ and $g$ are non negative measurable functions such that $f_n \to f$ a.e. and $\forall n$ and

$$\forall x \in E, f_n(x) \leq \left[ \frac{3n+1}{n+1} + \frac{2n+1}{n} \right] g(x).$$

Prove that $\lim_{\varepsilon \to 0} \int_E f dm \leq \int_E g dm$ ($m$ is Lebesgue measure).

7. Let $M$ be a closed linear subspace of a real normed linear space $N$, and let $x_0 \in N$ be a vector not in $M$. Set $d = \inf \{ \|x - x_0\| : x \in M \}$. Prove that there is a linear functional $f : N \to R$ such that $f(M) = 0$, $f(x_0) = 1$ and $\|f\| = \frac{1}{d}$.

8. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f_n \to f$ in $L^p(\mu)$ and $g_n \to g$ in $L^q(\mu)$, $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f_n g_n \to fg$ in $L^1(\mu)$. 