REAL ANALYSIS PRELIMINARY EXAMINATION SPRING 2005

Wednesday, January 5, 2005 9:00 a.m. - 12:00 noon

INSTRUCTIONS: Please choose six (6) of the following eight (8) problems.

Each solution you submit should be on a separate page (or pages). Be sure that your name and the problem number appear with each solution.

1) Let \((X, \mathcal{M}, \mu)\) be a measure space, and suppose that \(\mu(N) = 0\). Show that for any \(\mu\)-measurable function \(f\) we have that \(\int_N f \, d\mu = 0\).

2) Let \((X, \mathcal{M}, \mu)\) be a measure space and suppose \(\{f_n\}\) is a sequence of nonnegative measurable functions such that \(f_n \to f\) a.e. and such that \(\int_X f_n \, d\mu = \frac{2n+1}{5n+2}\). Prove that \(\int_X f \, d\mu < 1/2\).

3) Define the measure \(\nu\) on the Borel sets in \((0, 2\pi)\) by \(\nu(E) = \int_E \sin 2t \, dm(t)\) where \(m\) is Lebesgue measure.
   (a) Find the Jordan decomposition for \(\nu\).
   (b) Calculate \([\nu((0, 2\pi))\) and \([\nu((0, 2\pi))\)].

4) Show that \(\lim_{n \to \infty} \int_0^\infty \frac{1+2x^2}{x^2+1} \sin x + e^{-x} \, dx = 0\).

5) Let \(\mathcal{F}\) be a sub \(\sigma\)-algebra of the Borel sets \(B\) in \([0, 1]\) and let \(f\) be a \(B\)-measurable function such that \(E[f|\mathcal{F}] = 0\). Let \(g\) be a bounded \(\mathcal{F}\)-measurable function on \([0, 1]\). Prove that \(\int_0^1 fg \, dx = 0\).

6) Let \(H\) be a (separable) Hilbert space and let \(\{\phi_n\}_{n=1}^\infty\) be an orthonormal basis for \(H\). For \(f, g \in H\) let \(\alpha_n = (f, \phi_n)\) and \(\beta_n = (g, \phi_n)\). Prove that \((f, g) = \sum_{n=1}^\infty \alpha_n \beta_n\).

7) Let \(H\) be a Hilbert space and let \(\{x_n\}\) and \(\{y_n\}\) be two sequences in \(H\) such that \(x_n \to x\)
   and \(y_n \to y\). Show that \((x_n, y_n) \to (x, y)\).

8) Let \(A : H \to H\) be a bounded operator on a Hilbert space \(H\) such that there exists a real number \(K\) such that \(\|Ax\| \leq K\|x\|\) for all \(x \in H\).
   (a) Prove that \(A\) is one-to-one.
   (b) Prove that the range of \(A\) is closed.