Instructions:
You are to choose five (5) of the attached eight problems. Each problem is on a separate sheet. Please print your name on each of the eight sheets, and indicate (e.g. by writing "not chosen") those problems you do not choose.

Turn in the eight sheets separately.

Prelim. Exam '96
Real Analysis

Name ____________________________

1) Let $T$ be a closed linear mapping from $X$ to $Y$, $X$ and $Y$ being complete metric spaces. Assume that whenever the sequence $\{x_n; n \in \mathbb{N}\}$ converges in $X$, then $\{Tx_n; n \in \mathbb{N}\}$ is a Cauchy sequence in $Y$.
Show that $T$ is continuous.

2) Let $(X, \| \cdot \|)$ be a normed vector space. Show
a) for any $x \neq 0$ in $X$, there is a bounded linear functional $F \in X^*$ with $F(x) = \|x\|$.
   (Hint: Start with $M = \text{span} \{x\}$ and define first $f : M \rightarrow \mathbb{R}$ with $f(\lambda x) = \lambda \|x\|$, $\lambda \in \mathbb{R}$.)
   b) $X^*$ separates the points of $X$.

3) Let $T : (X, \| \cdot \|_X) \rightarrow (Y, \| \cdot \|_Y)$ be a linear mapping from the normed vector space $X$ into the normed vector space $Y$. Show that the following statements are equivalent
   (i) $T$ is continuous
   (ii) $T$ is continuous at 0
   (iii) $T$ is bounded
   (Hint: If $x \in X$, then $\frac{x}{\|x\|/\delta}$ has norm $\delta$ for $\delta > 0$)

4) Consider the measure space $(\mathbb{N}, P(\mathbb{N}), \nu)$, where $\mathbb{N}$ is the set of natural numbers, $P(\mathbb{N})$ the family of all subsets of $\mathbb{N}$ and $\nu$ the counting measure.
State the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem as statements about series.
5) Let \((X, T)\) be a topological space. Show that for any set \(A \subset X\):
\[(A^\circ)^c = \overline{A}^c,
\]
where \(A^\circ\) denotes the interior of \(A\), \(\overline{A}\) the closure of \(A\) and \(A^c = X\setminus A\) the complement of \(A\) in \(X\).

6) Let \((X, \mathcal{F}, \mu)\) be a measure space and \(f : X \to \mathbb{R}\) a non-negative \(\mathcal{F}\)-measurable function. Show
\[a) \lambda : \mathcal{F} \to \mathbb{R}\] defined by \(\lambda(A) = \int_A f \, d\mu\) is a measure on \(\mathcal{F}\); and
\[b) \text{for any } \mathcal{F}\text{-measurable } g \geq 0 \Rightarrow \int_X g d\lambda = \int_X g f d\mu.
\]

7) Let \((X, \mathcal{F}, \mu)\) be a measure space; let \(L^p = L^p(X, \mathcal{F}, \mu), p \geq 1\), and let \(1_A\) denote the characteristic (indicator) function of \(A \in \mathcal{F}\).
Show
\[a) \text{if } f \text{ and } g \text{ in } L^2, \text{ then } fg \in L^1; \text{ and}
\[b) \text{if } f \perp 1_A \text{ in } L_2 \text{ for all } A \in \mathcal{F} \text{ with } \mu(A) < \infty, \text{ then } f = 0 \mu \text{ a.e.}
\]

8) Given the functions \(f \in L^1(\mathbb{R}, B(\mathbb{R}), \text{Leb})\), \(\varphi : \mathbb{R} \to (0, \infty)\) continuous and
\[g_n = \frac{\varphi^{-1}(\frac{1}{n+1})}{\varphi^{-1}(\frac{1}{n-1})} f, \ n \in \mathbb{N},
\]
(where the root in the denominator is taken \(\geq 0\)), show that
\[
\int_{-\infty}^{\infty} g_n(x) \, dx \to \int_{-\infty}^{\infty} f(x) \, dx \quad (n \to \infty).
\]