Quadrature Formulas in Two Dimensions

Math 5172 - Finite Element Method
Section 001, Spring 2010
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When implementing FEM for solving two-dimensional partial differential equations, integrals of the form

$$I = \iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y$$

have to be evaluated to obtain local stiffness matrices and local load vectors, where K usually is an either quadrilateral or triangular element. Because the integrand generally depends on user-specified information, such as $p(x,y), q(x,y)$ and $f(x,y)$ in the second-order self-adjoint elliptic equation:

$$-\nabla \cdot (p(x,y)\nabla u) + q(x,y)u = f(x,y), \quad (x,y) \in \Omega,$$

in computer programs the analogous integrals shall be evaluated numerically.

A. Recall - Gaussian quadrature in one dimension

(1) **Gaussian quadrature of order N for the standard interval $I_{\mathrm{st}} = [-1,1]$:**

$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx \sum_{i=1}^{N} w_i g(\xi_i), \tag{1}$$

where ξ_i and w_i are Gaussian quadrature points and weights. Remember that a Gaussian quadrature using N points can provide the exact integral if $g(\xi)$ is a polynomial of degree $2N-1$ or less.

(i) Gaussian quadrature of order 1 (one point):

$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx 2g(0).$$

(ii) Gaussian quadrature of order 2 (two points):

$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right).$$

(iii) Gaussian quadrature of order 3 (three points):

$$\int_{-1}^{1} g(\xi)\ \mathrm{d}\xi \approx \frac{5}{9}\cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{8}{9}\cdot g\left(0\right) + \frac{5}{9}\cdot g\left(\frac{\sqrt{3}}{\sqrt{5}}\right).$$

(3) **Gaussian quadrature for general intervals $I = [a,b]$:**

$$x = a + \frac{b-a}{2}(1+\xi), \quad \text{or} \quad \xi = \frac{x-a}{b-a} + \frac{x-b}{b-a}$$

$$\Downarrow$$

$$\int_a^b F(x)\ \mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} F\left(a + \frac{b-a}{2}(1+\xi)\right)\mathrm{d}\xi.$$

Therefore, we have

$$\int_a^b F(x)\ \mathrm{d}x \approx \frac{b-a}{2}\sum_{i=1}^{N} w_i F\left(a + \frac{b-a}{2}(1+\xi_i)\right).$$

B. Gaussian quadrature for quadrilateral elements

(1) **Gaussian quadrature for the standard quadrilateral element** $R_{\text{st}} = [-1, 1]^2$**:**

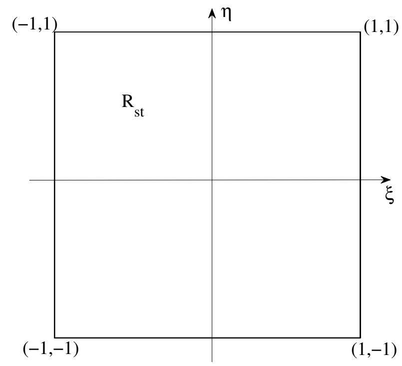


Figure 1: The standard quadrilateral element R_{st}.

$$I = \iint_{R_{\text{st}}} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta = \int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta.$$

For any fixed η, we can integrate numerically with respect to ξ:

$$\int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta \approx \int_{-1}^{1} \left(\sum_{i=1}^{M} w_i g(\xi_i, \eta) \right) \mathrm{d}\eta,$$

where ξ_i and w_i are Gaussian quadrature points and weights of order M in the ξ direction. Next integrating numerically with respect to η we have

$$\int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta \approx \sum_{i=1}^{M} \sum_{j=1}^{N} w_i \hat{w}_j g(\xi_i, \eta_j),$$

where η_j and $\hat{w}_j$ are Gaussian quadrature points and weights of order N in the η direction. This integration rule is exact if the integrand $g(\xi, \eta)$ contains only the monomials $\xi^i \eta^j (i = 0, 1, \cdots, 2M-1, j = 0, 1, \cdots, 2N-1)$. Usually $M = N$ (so $\eta_i = \xi_i, \hat{w}_i = w_i$), and we have

Gaussian quadrature of order N for the standard quadrilateral element

$$\int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta \approx \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j g(\xi_i, \xi_j). \tag{2}$$

Example: *Gaussian quadrature of order 3 for the standard quadrilateral element* $R_{\text{st}} = [-1, 1]^2$*:*

$$\begin{aligned} \int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta \approx & \frac{5}{9} \cdot \frac{5}{9} \cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}}, -\frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{8}{9} \cdot \frac{5}{9} \cdot g\left(0, -\frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{5}{9} \cdot \frac{5}{9} \cdot g\left(\frac{\sqrt{3}}{\sqrt{5}}, -\frac{\sqrt{3}}{\sqrt{5}}\right) \\ & + \frac{5}{9} \cdot \frac{8}{9} \cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}}, 0\right) + \frac{8}{9} \cdot \frac{8}{9} \cdot g\,(0, 0) + \frac{5}{9} \cdot \frac{8}{9} \cdot g\left(\frac{\sqrt{3}}{\sqrt{5}}, 0\right) \\ & + \frac{5}{9} \cdot \frac{5}{9} \cdot g\left(-\frac{\sqrt{3}}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{8}{9} \cdot \frac{5}{9} \cdot g\left(0, \frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{5}{9} \cdot \frac{5}{9} \cdot g\left(\frac{\sqrt{3}}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{5}}\right). \end{aligned}$$

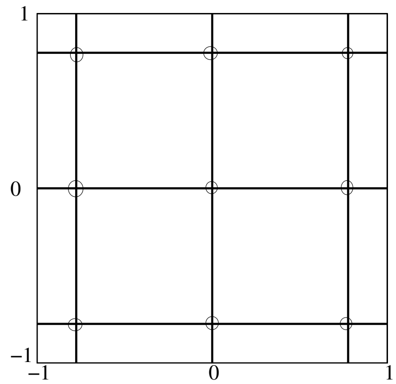


Figure 2: Gaussian quadrature points ($N = 3$) for the standard quadrilateral element $R_{\rm st} = [-1, 1]^2$.

(2) **Gaussian quadrature for general quadrilateral elements:**

- Let K be a quadrilateral element with straight boundary lines and vertices $(x_i, y_i), i = 1, 2, 3, 4$ arranged in the counter-clockwise order:

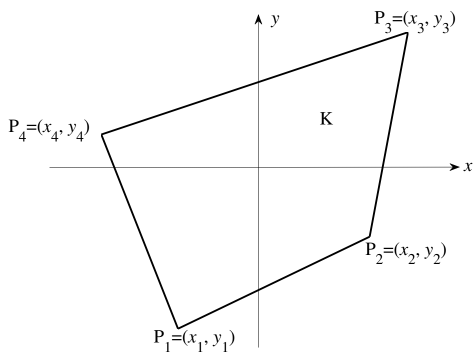


Figure 3: A quadrilateral element with straight boundary lines.

We would like to evaluate

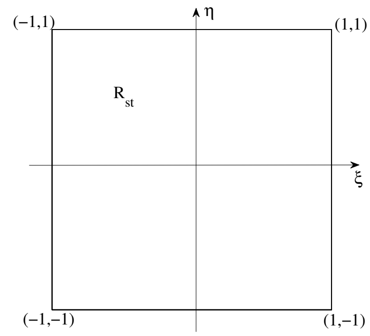
$$I = \iint_K F(x, y)\ {\rm d}x{\rm d}y.$$

- The idea is simple: first transform the quadrilateral element K to the standard quadrilateral element $R_{\rm st}$ and then apply the Gaussian quadrature (2).

How to transform: Using nodal shape functions!

- Nodal shape functions for quadrilaterals:

$$\begin{aligned} N_1(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 - \eta), \\ N_2(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 - \eta), \\ N_3(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 + \eta), \\ N_4(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 + \eta). \end{aligned}$$



Note $N_i(\xi, \eta) = 1$ at Node $\boxed{\rm i}$; 0 at other nodes.

- Construct a linear mapping to map the quadrilateral element K with straight boundary lines to the standard quadrilateral element $R_{\rm st}$:

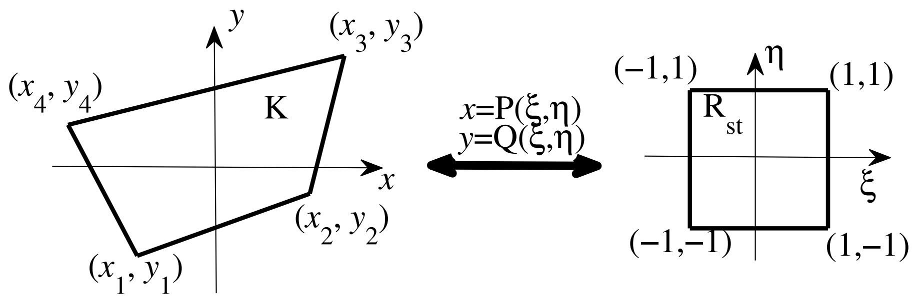


Figure 4: Linear mapping between K and $R_{\rm st}$

The mapping can be achieved conveniently by using the nodal shape functions as follows:

$$x = P(\xi,\eta) = \sum_{i=1}^{4} x_i N_i(\xi,\eta) = x_1 N_1(\xi,\eta) + x_2 N_2(\xi,\eta) + x_3 N_3(\xi,\eta) + x_4 N_4(\xi,\eta),$$
$$y = Q(\xi,\eta) = \sum_{i=1}^{4} y_i N_i(\xi,\eta) = y_1 N_1(\xi,\eta) + y_2 N_2(\xi,\eta) + y_3 N_3(\xi,\eta) + y_4 N_4(\xi,\eta).$$

Then we have

$$\iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y = \iint_{R_{\rm st}} F\left(P(\xi,\eta), Q(\xi,\eta)\right) |J(\xi,\eta)|\ \mathrm{d}\xi\mathrm{d}\eta,$$

where $J(\xi,\eta)$ is the Jacobian of the transformation defined by

$$J(\xi,\eta) = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}.$$

- Applying the Gaussian quadrature (2) for the standard quadrilateral element yields

Gaussian quadrature of order N for general quadrilateral elements

$$\iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j F(P(\xi_i,\xi_j), Q(\xi_i,\xi_j))|J(\xi_i,\xi_j)|.$$

C. Gaussian quadrature for triangular elements

(1) **Gaussian quadrature for the standard triangular element** $T_{\mathrm{st}} = \{(\xi, \eta) : 0 \le \xi, \eta, \xi + \eta \le 1\}$**:**

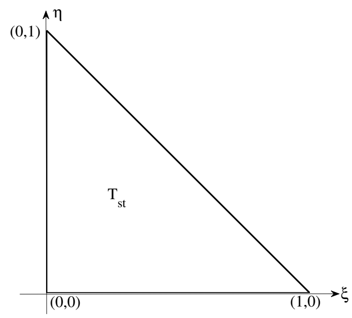


Figure 5: The standard triangular element T_{st}.

$$I = \iint_{T_{\mathrm{st}}} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta = \int_0^1 \int_0^{1-\eta} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta = \int_0^1 \int_0^{1-\xi} g(\xi, \eta)\ \mathrm{d}\eta \mathrm{d}\xi.$$

(a) **Tensor product-type Gaussian quadrature - Simple but less efficient:**

Idea: Transform the standard triangular element T_{st} to the standard quadrilateral element R_{st}, and then apply the Gaussian quadrature for R_{st}. Such transformation is defined by

$$\begin{cases} \xi = \dfrac{(1+a)(1-b)}{4}, \\ \eta = \dfrac{1+b}{2}, \end{cases} \quad \text{or} \quad \begin{cases} a = \dfrac{2\xi}{1-\eta} - 1, \\ b = 2\eta - 1. \end{cases} \tag{3}$$

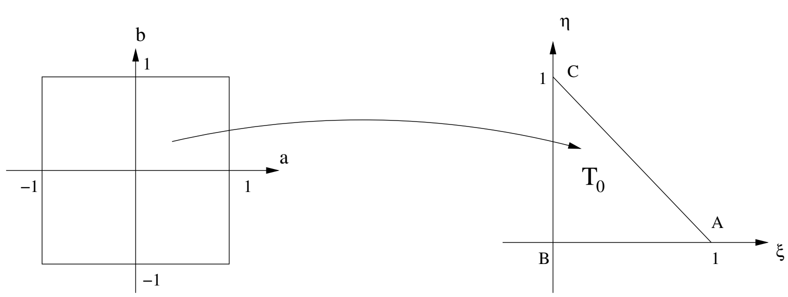


Figure 6: Illustration of the mapping between the square R_{st} and the triangle T_{st}.

The transformations defined in (3) basically collapse the top edge of the square R_{st} into the top vertex (0,1) of the triangle T_{st}. The Jacobians of the transformations are

$$J(a, b) = \left| \frac{\partial(\xi, \eta)}{\partial(a, b)} \right| = \frac{1-b}{8} = \frac{1-\eta}{4},$$

$$J^{-1}(\xi, \eta) = \left| \frac{\partial(a, b)}{\partial(\xi, \eta)} \right| = \frac{4}{1-\eta}.$$

Therefore, we have

$$I = \iint_{T_{\mathrm{st}}} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta = \iint_{R_{\mathrm{st}}} g(\xi(a, b), \eta(a, b)) |J(a, b)|\ \mathrm{d}a \mathrm{d}b.$$

Remark 1: Note that one direction of the transformation defined in (3) is linear and the corresponding Jacobian is bounded. However, the other one is non-linear, and its Jacobian has a singularity. Fortunately, this Jacobian is NOT used!

Remark 2: Tensor product-type quadrature have several advantages. In particular, their derivation and application is straightforward. They are versatile in that many one-dimensional rules are available for several different integrands. Extremely high-order polynomials may be evaluated, although precision may be limited since most references provide points and weights to 20 significant digits at most. The primary disadvantage is inefficiency since for high N, a relatively large number of points is required, and other quadrature rules are available with many fewer points. The secondary disadvantage is that the location of the points is unsymmetrical. Except for rules of low order, a large number of points will be concentrated in a relatively small region near one vertex (the top $(0,1)$ vertex). Such an arrangement, although correct, maybe considered aesthetically undesirable. For details, please read Ref. [2].

(b) **Symmetrical Gaussian quadrature on triangles:**

- **Goal:** The goal is to develop quadrature rules of the form

$$\iint_{T_{\rm st}} g(\xi,\eta)\ {\rm d}\xi{\rm d}\eta \approx \frac{1}{2}\sum_{i=1}^{N_g} w_i g(\xi_i,\eta_i), \tag{4}$$

where N_g is the number of quadrature points, (ξ_i,η_i) are quadrature points located inside the standard triangle and w_i are weights (normalized with respect to the triangle area). In addition to the criteria that the resulting quadrature should use as less as possible number of quadrature points to achieve as high as possible accuracy, we also would like the quadrature points possess some kind of symmetry.

- **Gaussian quadrature of degree N for triangles:**

The basic idea is the same as that used in developing Gaussian quadrature for the standard interval $I_{\rm st}=[-1,1]$. We want to choose points (ξ_i,η_i) and weights w_i in (4) so that the quadrature (4) is as accurate as possible in some sense. Generally speaking, a **Gaussian quadrature of degree N for triangles** is defined as a quadrature of (4) that is exact for arbitrary polynomial of degree N, namely,

$$\iint_{T_{\rm st}} g(\xi,\eta)\ {\rm d}\xi{\rm d}\eta = \frac{1}{2}\sum_{i=1}^{N_g} w_i g(\xi_i,\eta_i), \qquad \forall g(\xi,\eta)\in \mathbf{P}_N(\xi,\eta),$$

where $\mathbf{P}_N(\xi,\eta)$ represents the complete polynomial space of degree N in two dimensions

$$\mathbf{P}_N(\xi,\eta) = \textbf{span}\left\{\xi^i\eta^j,\ \ 0\le i,j,i+j\le N\right\}.$$

For examples,

$$\mathbf{P}_1(\xi,\eta) = \textbf{span}\left\{1,\xi,\eta\right\},$$
$$\mathbf{P}_2(\xi,\eta) = \textbf{span}\left\{1,\xi,\eta,\xi^2,\xi\eta,\eta^2\right\}.$$

Note that

$$\dim(\mathbf{P}_N) = \frac{(N+1)(N+2)}{2}.$$

- A useful identity:

$$\iint_{T_{\rm st}} \xi^i\eta^j\ {\rm d}\xi{\rm d}\eta = \frac{i!j!}{(i+j+2)!}.$$

Therefore, we have

$$\iint_{T_{\rm st}} \left\{1,\xi,\eta,\xi^2,\xi\eta,\eta^2,\xi^3,\xi^2\eta,\xi\eta^2,\eta^3\right\}\ {\rm d}\xi{\rm d}\eta = \left\{\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{12},\frac{1}{24},\frac{1}{12},\frac{1}{20},\frac{1}{60},\frac{1}{60},\frac{1}{20}\right\}.$$

- **Gaussian quadrature of degree 1 for triangles:**

  By definition, the quadrature should be accurate for $g(\xi,\eta) = 1, \xi$, and η. We get

$$g(\xi,\eta) = 1 \quad \longrightarrow \quad \frac{1}{2} = \frac{1}{2}\sum_{i=1}^{N_g} w_i$$

$$g(\xi,\eta) = \xi \quad \longrightarrow \quad \frac{1}{6} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i$$

$$g(\xi,\eta) = \eta \quad \longrightarrow \quad \frac{1}{6} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\eta_i.$$

  It is easy to see that $N_g = 1$, $w_1 = 1$ and $\xi_1 = \eta_1 = \dfrac{1}{3}$ is a solution. Therefore, we have

  $\boxed{\text{Gaussian quadrature of degree 1 for the standard triangle } T_{\text{st}}}$

$$\iint_{T_{\text{st}}} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta = \frac{1}{2}g\left(\frac{1}{3},\frac{1}{3}\right).$$

- **Gaussian quadrature of degree 2 for triangles:**

  By definition, the quadrature should be accurate for $g(\xi,\eta) = 1, \xi, \eta, \xi^2, \xi\eta$, and η^2. We get

$$g(\xi,\eta) = 1 \quad \longrightarrow \quad \frac{1}{2} = \frac{1}{2}\sum_{i=1}^{N_g} w_i$$

$$g(\xi,\eta) = \xi \quad \longrightarrow \quad \frac{1}{6} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i$$

$$g(\xi,\eta) = \eta \quad \longrightarrow \quad \frac{1}{6} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\eta_i$$

$$g(\xi,\eta) = \xi^2 \quad \longrightarrow \quad \frac{1}{12} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i^2$$

$$g(\xi,\eta) = \xi\eta \quad \longrightarrow \quad \frac{1}{24} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i\eta_i$$

$$g(\xi,\eta) = \eta^2 \quad \longrightarrow \quad \frac{1}{12} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\eta_i^2.$$

  It is obvious that $N_g = 1$ will NOT work since we have 6 equations. In theory, $N_g = 2$ may work since in this case we are going to have 6 unknowns $(\xi_1, \eta_1, \xi_2, \eta_2, w_1, w_2)$, but the resulting quadrature is NOT symmetric! So we choose $N_g = 3$. Using $N_g = 3$ (three points), we have 9 unknowns. Because there are only 6 equations, generally speaking the solution is not unique. We can verify that

$$(\xi_1,\eta_1) = \left(\frac{1}{6},\frac{1}{6}\right),\ (\xi_2,\eta_2) = \left(\frac{2}{3},\frac{1}{6}\right),\ (\xi_3,\eta_3) = \left(\frac{1}{6},\frac{2}{3}\right),\ w_1 = w_2 = w_3 = \frac{1}{3}$$

  is a solution. Therefore, we have

  $\boxed{\text{Gaussian quadrature of degree 2 for the standard triangle } T_{\text{st}}}$

$$\iint_{T_{\text{st}}} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta = \frac{1}{6}\left[g\left(\frac{1}{6},\frac{1}{6}\right) + g\left(\frac{2}{3},\frac{1}{6}\right) + g\left(\frac{1}{6},\frac{2}{3}\right)\right].$$

Remember that the solution is NOT unique! For example, we can also verify that

$$(\xi_1, \eta_1) = \left(0, \frac{1}{2}\right), \ (\xi_2, \eta_2) = \left(\frac{1}{2}, 0\right), \ (\xi_3, \eta_3) = \left(\frac{1}{2}, \frac{1}{2}\right), \ w_1 = w_2 = w_3 = \frac{1}{3}$$

is also a solution. Therefore, we have another

$\boxed{\text{Gaussian quadrature of degree 2 for the standard triangle } T_{\text{st}}}$

$$\iint_{T_{\text{st}}} g(\xi, \eta)\ \mathrm{d}\xi\mathrm{d}\eta = \frac{1}{6}\left[g\left(0, \frac{1}{2}\right) + g\left(\frac{1}{2}, 0\right) + g\left(\frac{1}{2}, \frac{1}{2}\right)\right].$$

- **Gaussian quadrature of degree 3 for triangles:**

By definition, the quadrature shall be accurate for $g(\xi, \eta) = 1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2$, and η^3. So

$$\begin{aligned}
g(\xi, \eta) = 1 \quad &\longrightarrow \quad \frac{1}{2} = \frac{1}{2}\sum_{i=1}^{N_g} w_i \\
g(\xi, \eta) = \xi \quad &\longrightarrow \quad \frac{1}{6} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i \\
g(\xi, \eta) = \eta \quad &\longrightarrow \quad \frac{1}{6} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\eta_i \\
g(\xi, \eta) = \xi^2 \quad &\longrightarrow \quad \frac{1}{12} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i^2 \\
g(\xi, \eta) = \xi\eta \quad &\longrightarrow \quad \frac{1}{24} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i\eta_i \\
g(\xi, \eta) = \eta^2 \quad &\longrightarrow \quad \frac{1}{12} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\eta_i^2 \\
g(\xi, \eta) = \xi^3 \quad &\longrightarrow \quad \frac{1}{20} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i^3 \\
g(\xi, \eta) = \xi^2\eta \quad &\longrightarrow \quad \frac{1}{60} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i^2\eta_i \\
g(\xi, \eta) = \xi\eta^2 \quad &\longrightarrow \quad \frac{1}{60} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\xi_i\eta_i^2 \\
g(\xi, \eta) = \eta^3 \quad &\longrightarrow \quad \frac{1}{20} = \frac{1}{2}\sum_{i=1}^{N_g} w_i\eta_i^3.
\end{aligned}$$

It is obvious that $N_g = 3$ will NOT work since we have 10 equations. So we choose $N_g = 4$. Using $N_g = 4$ (four points), we have 12 unknowns. Because there are only 10 equations, again generally speaking the solution is not unique. We can verify that

$$(\xi_1, \eta_1) = \left(\frac{1}{3}, \frac{1}{3}\right), \ (\xi_2, \eta_2) = \left(\frac{1}{5}, \frac{1}{5}\right), \ (\xi_3, \eta_3) = \left(\frac{1}{5}, \frac{3}{5}\right), \ (\xi_4, \eta_4) = \left(\frac{3}{5}, \frac{1}{5}\right),$$

$$w_1 = -\frac{27}{48}, \quad w_2 = w_3 = w_4 = \frac{25}{48}$$

is a solution. Therefore, we have

$$\boxed{\text{Gaussian quadrature of degree 3 for the standard triangle } T_{\rm st}}$$

$$\iint_{T_{\rm st}} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta = -\frac{27}{96}\cdot g\left(\frac{1}{3},\frac{1}{3}\right) + \frac{25}{96}\left[g\left(\frac{1}{5},\frac{1}{5}\right) + g\left(\frac{1}{5},\frac{3}{5}\right) + g\left(\frac{3}{5},\frac{1}{5}\right)\right].$$

Again recall that the solution is NOT unique! For example, we can verify that

$$(\xi_1,\eta_1) = \left(\frac{1}{3},\frac{1}{3}\right),\ (\xi_2,\eta_2) = \left(\frac{2}{15},\frac{11}{15}\right),\ (\xi_3,\eta_3) = \left(\frac{2}{15},\frac{2}{15}\right),\ (\xi_4,\eta_4) = \left(\frac{11}{15},\frac{2}{15}\right),$$

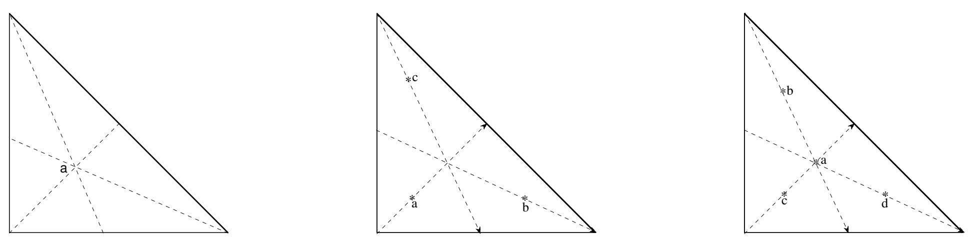
$$w_1 = -\frac{27}{48},\quad w_2 = w_3 = w_4 = \frac{25}{48}$$

is also a solution. Therefore, we have another

$$\boxed{\text{Gaussian quadrature of degree 3 for the standard triangle } T_{\rm st}}$$

$$\iint_{T_{\rm st}} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta = -\frac{27}{96}\cdot g\left(\frac{1}{3},\frac{1}{3}\right) + \frac{25}{96}\left[g\left(\frac{2}{15},\frac{11}{15}\right) + g\left(\frac{2}{15},\frac{2}{15}\right) + g\left(\frac{11}{15},\frac{2}{15}\right)\right].$$

- **Graphical illustration of quadrature points:**

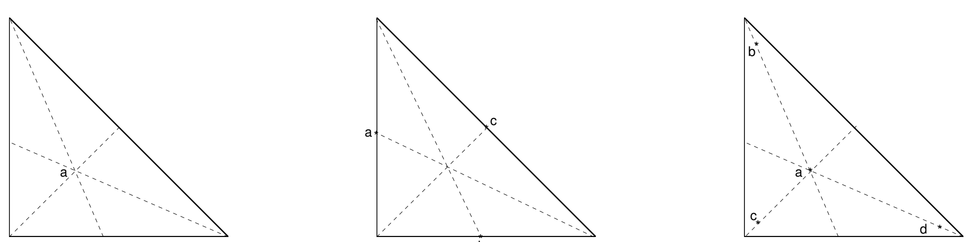


(a) Linear
$a = \left(\frac{1}{3},\frac{1}{3}\right),\ w = 1$

(b) Quadratic
$a = \left(\frac{1}{6},\frac{1}{6}\right),\ w = \frac{1}{3}$
$b = \left(\frac{2}{3},\frac{1}{6}\right),\ w = \frac{1}{3}$
$c = \left(\frac{1}{6},\frac{2}{3}\right),\ w = \frac{1}{3}$

(c) Cubic
$a = \left(\frac{1}{3},\frac{1}{3}\right),\ w = -\frac{27}{48}$
$b = \left(\frac{1}{5},\frac{3}{5}\right),\ w = \frac{25}{48}$
$c = \left(\frac{1}{5},\frac{1}{5}\right),\ w = \frac{25}{48}$
$d = \left(\frac{3}{5},\frac{1}{5}\right),\ w = \frac{25}{48}$

Figure 7: First set of quadrature rules for triangular elements



(a) Linear
$a = \left(\frac{1}{3},\frac{1}{3}\right),\ w = 1$

(b) Quadratic
$a = \left(0,\frac{1}{2}\right),\ w = \frac{1}{3}$
$b = \left(\frac{1}{2},0\right),\ w = \frac{1}{3}$
$c = \left(\frac{1}{2},\frac{1}{2}\right),\ w = \frac{1}{3}$

(c) Cubic
$a = \left(\frac{1}{3},\frac{1}{3}\right),\ w = -\frac{27}{48}$
$b = \left(\frac{2}{15},\frac{11}{15}\right),\ w = \frac{25}{48}$
$c = \left(\frac{2}{15},\frac{2}{15}\right),\ w = \frac{25}{48}$
$d = \left(\frac{11}{15},\frac{2}{15}\right),\ w = \frac{25}{48}$

Figure 8: Second set of quadrature rules for triangular elements

- **Number of quadrature points N_g:**

  The most commonly referenced Gauss-Legendre locations and weights for triangles are the symmetric quadrature rules of [1]. This reference provides tables varying from degrees 1 to 20 (79 quadrature points), which are reproduced (for degrees between 1 and 12) in **2DTriGaussPoints.m**. The weights in these tables are normalized with respect to triangle area, i.e.,

$$\iint_{T_{\text{st}}} g(\xi, \eta)\ \mathrm{d}\xi \mathrm{d}\eta \approx \frac{1}{2} \sum_{i=1}^{N_g} w_i g(\xi_i, \eta_i).$$

  The following tables list the number of quadrature points for degrees 1 to 20 as given in Ref. [1]. It should be mentioned that for some N, the corresponding N_g is not necessarily unique. If interested, please see [1] and references therein.

N	1	2	3	4	5	6	7	8	9	10	11	12
$\dim(\mathbf{P}_N)$	3	6	10	15	21	28	36	45	55	66	78	91
N_g	1	3	4	6	7	12	13	16	19	25	27	33

N	13	14	15	16	17	18	19	20
$\dim(\mathbf{P}_N)$	105	120	136	153	171	190	210	231
N_g	37	42	48	52	61	70	73	79

(2) **Gaussian quadrature for general triangular elements K:**

- Let K be a triangular element with straight boundary lines and vertices $(x_i, y_i), i = 1, 2, 3$ arranged in the counter-clockwise order:

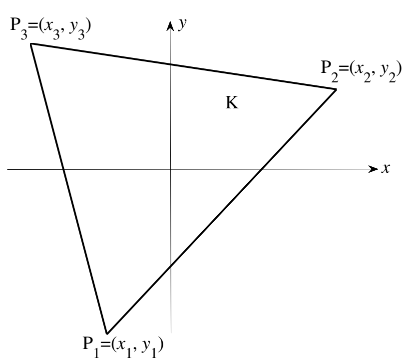


Figure 9: A triangular element with straight boundary lines.

  We would like to evaluate

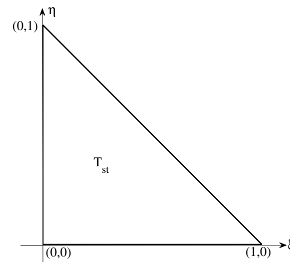
$$I = \iint_K F(x, y)\ \mathrm{d}x\mathrm{d}y,$$

- Again the idea is very simple: first transform the triangular element K to the standard triangular element T_{st} and then apply the Gaussian quadrature for T_{st} as described above.

  How to transform: Using nodal shape functions!

- Nodal shape functions for triangles:

$$N_1(\xi,\eta) = 1 - \xi - \eta,$$
$$N_2(\xi,\eta) = \xi,$$
$$N_3(\xi,\eta) = \eta.$$



- Construct a linear mapping to map the general triangular element K with straight boundary lines to the standard triangular element T_{st}:

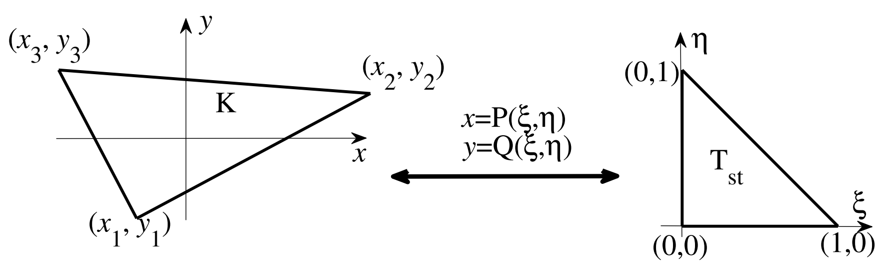


Figure 10: Linear mapping between K and T_{st}.

The mapping can be achieved conveniently by using the nodal shape functions as follows:

$$x = P(\xi,\eta) = \sum_{i=1}^{3} x_i N_i(\xi,\eta) = x_1 N_1(\xi,\eta) + x_2 N_2(\xi,\eta) + x_3 N_3(\xi,\eta),$$
$$y = Q(\xi,\eta) = \sum_{i=1}^{3} y_i N_i(\xi,\eta) = y_1 N_1(\xi,\eta) + y_2 N_2(\xi,\eta) + y_3 N_3(\xi,\eta).$$

Then we have

$$\iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y = \iint_{T_{\mathrm{st}}} F(P(\xi,\eta), Q(\xi,\eta))|J(\xi,\eta)|\ \mathrm{d}\xi\mathrm{d}\eta,$$

where $J(\xi,\eta)$ is the Jacobian of the transformation, namely,

$$J(\xi,\eta) = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = 2A_K.$$

Here A_K represents the area of the triangle K, which can be evaluated by

$$A_K = \frac{|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|}{2}.$$

Therefore, we have

$$\iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y = 2A_k \iint_{T_{\mathrm{st}}} F(P(\xi,\eta), Q(\xi,\eta))\ \mathrm{d}\xi\mathrm{d}\eta.$$

- Applying the Gaussian quadrature of degree N for the standard triangular element (4) yields

$$\boxed{\textbf{Gaussian quadrature of degree } N \textbf{ for general triangular elements}}$$

$$\iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y \approx A_K \sum_{i=1}^{N_g} w_i F(P(\xi_i,\eta_i), Q(\xi_i,\eta_i)). \tag{5}$$

- Note that the inverse of the transformation shown in Fig. 10 is NOT needed, but it can be found that

$$\begin{aligned}\xi &= \frac{1}{2A_K}\left[(y_3-y_1)(x-x_1)-(x_3-x_1)(y-y_1)\right],\\ \eta &= \frac{1}{2A_K}\left[-(y_2-y_1)(x-x_1)+(x_2-x_1)(y-y_1)\right].\end{aligned}$$

(3) **Implementation**: The Matlab code using the Gaussian quadrature (5) to evaluate

$$I = \iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y$$

is **int_f.m**. It uses **f.m** which defines the function $F(x,y)$ and **TriGaussPoints.m** which provides Gaussian points and weights. The calling sequence of **int_f.m** is

```
int_f(quadrature_degree, x1, x2, x3, y1, y2, y3)
```

where (x_1,y_1), (x_2,y_2), and (x_3,y_3) are the coordinates of the three vertices of the triangular element K.

A simple test: Consider the following integral

$$I = \iint_K (2-x-2y)\ \mathrm{d}x\mathrm{d}y = \frac{1}{3},$$

where K is the triangle defined by the three vertices: $(0,0),(1,1/2)$ and $(0,1)$. Since $f(x,y)=2-x-2y$ has degree of 1, Gaussian quadrature with $N=2$ should be able to provide the exact integral. Actually if we run

```
int_f(2, 0, 1, 0, 0, 0.5, 1)
```

in Matlab, it returns 0.333333333333323, which is 1/3.

D. Contour integrals

- In many cases (such as when natural or mixed boundary conditions are involved) we have to evaluate integrals along the boundary of a (boundary) element which are of the form:

$$I = \int_{P_i}^{P_j} B(x,y)\ \mathrm{d}s,$$

where P_i and P_j are consecutive nodal points in the counter-clockwise order and $\mathrm{d}s = \sqrt{\mathrm{d}x^2+\mathrm{d}y^2}$ is the differential arc length along the boundary of the element.

The idea is: first transform the straight contour P_iP_j to an interval $I=[a,b]$, and then apply the Gaussian quadrature for the interval.

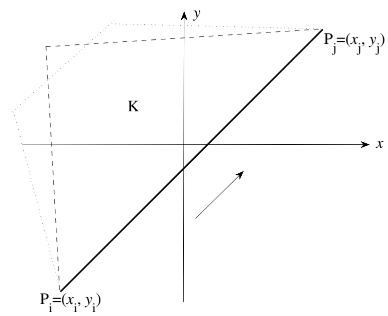


Figure 11: A (straight) contour.

Suppose that the following transformation is used to transform a general quadrilateral/triangular element K to the standard quadrilateral/triangular element R_{st} or T_{st}:

$$x = P(\xi, \eta), \qquad y = Q(\xi, \eta).$$

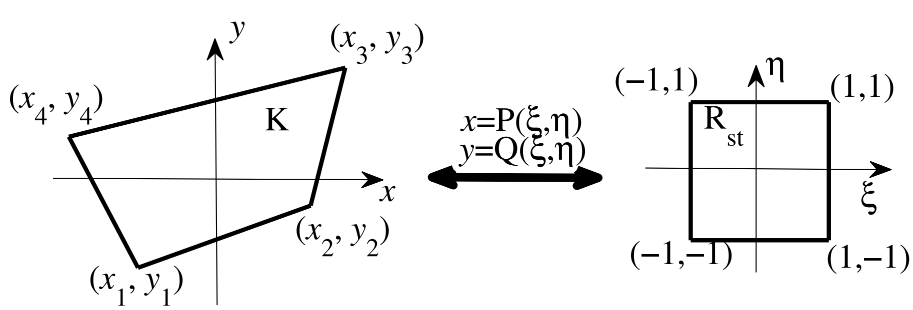
Then, we have

$$\mathrm{d}x = \frac{\partial x}{\partial \xi}\mathrm{d}\xi + \frac{\partial x}{\partial \eta}\mathrm{d}\eta = J_{11}\mathrm{d}\xi + J_{21}\mathrm{d}\eta, \tag{6a}$$

$$\mathrm{d}y = \frac{\partial y}{\partial \xi}\mathrm{d}\xi + \frac{\partial y}{\partial \eta}\mathrm{d}\eta = J_{12}\mathrm{d}\xi + J_{22}\mathrm{d}\eta, \tag{6b}$$

where J_{11}, J_{12}, etc, are elements of the Jacobian matrix $J(\xi, \eta)$.

- **The contour P_iP_j is a boundary line of a quadrilateral element:**



Along each side of a quadrilateral element, either ξ or η is fixed. For example, along side 1 (P_1P_2), $\eta = -1$ so $\mathrm{d}\eta = 0$. In this case, we have

$$\mathrm{d}x = \left(\frac{\partial x}{\partial \xi}\right)_{\eta=-1}\mathrm{d}\xi = J_{11}(\xi, -1)\mathrm{d}\xi,$$

$$\mathrm{d}y = \left(\frac{\partial y}{\partial \xi}\right)_{\eta=-1}\mathrm{d}\xi = J_{12}(\xi, -1)\mathrm{d}\xi.$$

Therefore,

$$\mathrm{d}s = \sqrt{J_{11}^2(\xi, -1) + J_{12}^2(\xi, -1)}\ \mathrm{d}\xi,$$

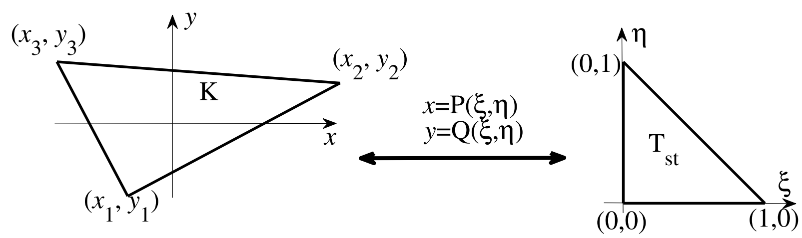
and accordingly, we have

$$\int_{P_1}^{P_2} B(x,y)\ \mathrm{d}s = \int_{-1}^{1} B(P(\xi,-1),Q(\xi,-1))\sqrt{J_{11}^2(\xi,-1)+J_{12}^2(\xi,-1)}\ \mathrm{d}\xi.$$

Similarly, we have

$$\int_{P_2}^{P_3} B(x,y)\ \mathrm{d}s = \int_{-1}^{1} B(P(1,\eta),Q(1,\eta))\sqrt{J_{21}^2(1,\eta)+J_{22}^2(1,\eta)}\ \mathrm{d}\eta,$$
$$\int_{P_3}^{P_4} B(x,y)\ \mathrm{d}s = -\int_{-1}^{1} B(P(\xi,1),Q(\xi,1))\sqrt{J_{11}^2(\xi,1)+J_{12}^2(\xi,1)}\ \mathrm{d}\xi,$$
$$\int_{P_4}^{P_1} B(x,y)\ \mathrm{d}s = -\int_{-1}^{1} B(P(-1,\eta),Q(-1,\eta))\sqrt{J_{21}^2(-1,\eta)+J_{22}^2(-1,\eta)}\ \mathrm{d}\eta.$$

- **The contour P_iP_j is a boundary line of a triangular element:**



Along side 1 (P_1P_2) or 3 (P_3P_1) of a triangular element, either ξ or η is fixed. Similarly, we have

$$\int_{P_1}^{P_2} B(x,y)\ \mathrm{d}s = \int_{0}^{1} B(P(\xi,0),Q(\xi,0))\sqrt{J_{11}^2(\xi,0)+J_{12}^2(\xi,0)}\ \mathrm{d}\xi,$$
$$\int_{P_3}^{P_1} B(x,y)\ \mathrm{d}s = -\int_{0}^{1} B(P(0,\eta),Q(0,\eta))\sqrt{J_{21}^2(0,\eta)+J_{22}^2(0,\eta)}\ \mathrm{d}\eta.$$

On the other hand, along side 2 (P_2P_3), $\xi+\eta=1$ so $\mathrm{d}\xi=-\mathrm{d}\eta$. In this case, we have

$$\mathrm{d}x = J_{11}\mathrm{d}\xi + J_{21}\mathrm{d}\eta = [J_{21}(1-\eta,\eta)-J_{11}(1-\eta,\eta)]\,\mathrm{d}\eta,$$
$$\mathrm{d}y = J_{12}\mathrm{d}\xi + J_{22}\mathrm{d}\eta = [J_{22}(1-\eta,\eta)-J_{12}(1-\eta,\eta)]\,\mathrm{d}\eta.$$

Therefore,

$$\mathrm{d}s = \sqrt{[J_{21}(1-\eta,\eta)-J_{11}(1-\eta,\eta)]^2+[J_{22}(1-\eta,\eta)-J_{12}(1-\eta,\eta)]^2}\ \mathrm{d}\eta = H(\eta)\mathrm{d}\eta,$$

and accordingly, we have

$$\int_{P_2}^{P_3} B(x,y)\ \mathrm{d}s = \int_{0}^{1} B(P(1-\eta,\eta),Q(-\eta,\eta))H(\eta)\ \mathrm{d}\eta.$$

Appendix A - Simplex coordinates

- The Gaussian quadrature for triangles described above can be applied to integrals over any triangular domain T in either two or three dimensions:

$$I = \iint_T F(\mathbf{r})\ \mathrm{d}\mathbf{r}.$$

In three dimensions, however, it is more convenient to consider these integrals in terms of *simplex coordinates*, also called *area coordinates* or *barycentric coordinates*.

- To develop the simplex coordinate transformation, consider the triangle T defined by the vertices $\mathbf{v}_1 = (x_1, y_1, z_1)$, $\mathbf{v}_2 = (x_2, y_2, z_2)$, and $\mathbf{v}_3 = (x_3, y_3, z_3)$ and the edges $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{e}_2 = \mathbf{v}_3 - \mathbf{v}_2$, and $\mathbf{e}_3 = \mathbf{v}_3 - \mathbf{v}_1$. Any point $\mathbf{r}$ located on the triangle can be written as a weighted sum of these three nodes via

$$\mathbf{r} = \gamma\mathbf{v}_1 + \alpha\mathbf{v}_2 + \beta\mathbf{v}_3, \tag{7}$$

where α, β, and γ are the simplex coordinates defined by

$$\alpha = \frac{A_1}{A}, \qquad \beta = \frac{A_2}{A}, \qquad \gamma = \frac{A_3}{A},$$

and A is the area of T. These coordinates are subject to the constraint

$$\alpha + \beta + \gamma = 1.$$

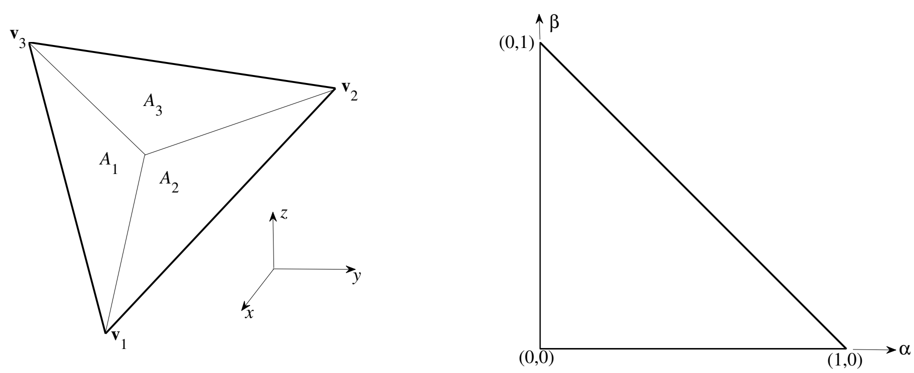
Therefore,

$$\gamma = 1 - \alpha - \beta$$

and

$$\mathbf{r} = (1 - \alpha - \beta)\mathbf{v}_1 + \alpha\mathbf{v}_2 + \beta\mathbf{v}_3.$$

Note that (7) has the form of a linear interpolation. Indeed, simplex coordinates will also allow us to perform a linear interpolation of a function $F(x, y)$ at points inside the triangle if its values are known at the vertices.



(a) The original triangle

(b) The transformed triangle

Figure 12: Simplex coordinates for a triangle.

The resulting transformation of the integral to simplex coordinates is

$$\iint_T F(\mathbf{r})\ \mathrm{d}\mathbf{r} = \iint_{T_{\mathrm{st}}} F(\alpha, \beta)|J(\alpha, \beta)|\ \mathrm{d}\alpha\mathrm{d}\beta.$$

Again, it can be proved that the Jacobian is

$$|J(\alpha, \beta)| = 2A,$$

where A is the area of the triangle T, which can be evaluated by

$$A = \frac{1}{2}\left|(\mathbf{v}_1 - \mathbf{v}_3) \times (\mathbf{v}_2 - \mathbf{v}_3)\right|.$$

Finally, we have

$$\iint_T F(\mathbf{r})\ \mathrm{d}\mathbf{r} = 2A \iint_{T_{\mathrm{st}}} F(\alpha,\beta)\ \mathrm{d}\alpha\mathrm{d}\beta = 2A \int_0^1 \int_0^{1-\alpha} F(\alpha,\beta)\ \mathrm{d}\beta\mathrm{d}\alpha.$$

- **From (x, y, z) to simplex coordinates:** Given a point $\mathbf{r} = (x, y, z)$ inside a triangle, sometimes it is also desirable to obtain the simplex coordinates (α, β, γ). We can write the barycentric expansion of $\mathbf{r} = (x, y, z)$ in terms of the components of the triangle vertices as

$$\begin{aligned} x &= \gamma x_1 + \alpha x_2 + \beta x_3, \\ y &= \gamma y_1 + \alpha y_2 + \beta y_3, \\ z &= \gamma z_1 + \alpha z_2 + \beta z_3. \end{aligned}$$

Substituting $\gamma = 1 - \alpha - \beta$ into the above gives

$$\begin{aligned} x &= (1-\alpha-\beta)x_1 + \alpha x_2 + \beta x_3, \\ y &= (1-\alpha-\beta)y_1 + \alpha y_2 + \beta y_3, \\ z &= (1-\alpha-\beta)z_1 + \alpha z_2 + \beta z_3. \end{aligned}$$

Rearranging, these are

$$\begin{aligned} \alpha(x_2 - x_1) + \beta(x_3 - x_1) + x_1 - x = 0, \\ \alpha(y_2 - y_1) + \beta(y_3 - y_1) + y_1 - y = 0, \\ \alpha(z_2 - z_1) + \beta(z_3 - z_1) + z_1 - z = 0. \end{aligned}$$

Solving for α and β gives us

$$\alpha = \frac{B(F+I) - C(E+H)}{A(E+H) - B(D+G)}$$

and

$$\beta = \frac{A(F+I) - C(D+G)}{B(D+G) - A(E+H)},$$

where

$$\begin{aligned} A &= x_2 - x_1, \\ B &= x_3 - x_1, \\ C &= x_1 - x, \\ D &= y_2 - y_1, \\ E &= y_3 - y_1, \\ F &= y_1 - y, \\ G &= z_2 - z_1, \\ H &= z_3 - z_1, \\ I &= z_1 - z. \end{aligned}$$

Appendix B - Matlab code of Gaussian quadrature for triangles

(1) **Main routine: int_f.m**

```
    function z = int_f(N,x1,x2,x3,y1,y2,y3)

%   This function evaluates \iint_K f(x,y) dxdy using
%   the Gaussian quadrature of order N where K is a
%   triangle with vertices (x1,y1), (x2,y2) and (x3,y3).

    xw = TriGaussPoints(N);  % get quadrature points and weights

    % calculate the area of the triangle
    A=abs(x1*(y2-y3)+x2*(y3-y1)+x3*(y1-y2))/2.0

    % find number of Gauss points
    NP=length(xw(:,1));

    z = 0.0;
    for j = 1:NP
        x = x1*(1-xw(j,1)-xw(j,2))+x2*xw(j,1)+x3*xw(j,2)
        y = y1*(1-xw(j,1)-xw(j,2))+y2*xw(j,1)+y3*xw(j,2)
        z = z + f(x,y)*xw(j,3);
    end
    z = A*z;

    return
    end
```

(2) **Function to define $F(x,y)$: f.m**

```
    function z=f(x,y)
    z=2-x-2*y;
    return;
    end
```

(3) **Function to provide Gaussian points (ξ_i, η_i) and weights w_i: TriGaussPoints.m**

```
  function xw = TriGaussPoints(n)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Function TriGaussPoints provides the Gaussian points and weights    %
%   for the Gaussian quadrature of order n for the standard triangles.  %
%                                                                       %
%   Input: n   - the order of the Gaussian quadrature (n<=12)          %
%                                                                       %
%   Output: xw - a n by 3 matrix:                                       %
%               1st column gives the x-coordinates of points            %
%               2nd column gives the y-coordinates of points            %
%               3rd column gives the weights                            %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
xw = zeros(n,3);

if (n == 1)
   xw=[0.33333333333333    0.33333333333333    1.00000000000000];

elseif (n == 2)
   xw=[0.16666666666667    0.16666666666667    0.33333333333333
       0.16666666666667    0.66666666666667    0.33333333333333
       0.66666666666667    0.16666666666667    0.33333333333333];

elseif (n == 3)
   xw=[0.33333333333333    0.33333333333333   -0.56250000000000
       0.20000000000000    0.20000000000000    0.52083333333333
       0.20000000000000    0.60000000000000    0.52083333333333
       0.60000000000000    0.20000000000000    0.52083333333333];

elseif (n == 4)
   xw=[0.44594849091597    0.44594849091597    0.22338158967801
       0.44594849091597    0.10810301816807    0.22338158967801
       0.10810301816807    0.44594849091597    0.22338158967801
       0.09157621350977    0.09157621350977    0.10995174365532
       0.09157621350977    0.81684757298046    0.10995174365532
       0.81684757298046    0.09157621350977    0.10995174365532];

elseif (n == 5)
   xw=[0.33333333333333    0.33333333333333    0.22500000000000
       0.47014206410511    0.47014206410511    0.13239415278851
       0.47014206410511    0.05971587178977    0.13239415278851
       0.05971587178977    0.47014206410511    0.13239415278851
       0.10128650732346    0.10128650732346    0.12593918054483
       0.10128650732346    0.79742698535309    0.12593918054483
       0.79742698535309    0.10128650732346    0.12593918054483];

elseif (n == 6)
   xw=[0.24928674517091    0.24928674517091    0.11678627572638
       0.24928674517091    0.50142650965818    0.11678627572638
       0.50142650965818    0.24928674517091    0.11678627572638
       0.06308901449150    0.06308901449150    0.05084490637021
       0.06308901449150    0.87382197101700    0.05084490637021
       0.87382197101700    0.06308901449150    0.05084490637021
       0.31035245103378    0.63650249912140    0.08285107561837
       0.63650249912140    0.05314504984482    0.08285107561837
       0.05314504984482    0.31035245103378    0.08285107561837
       0.63650249912140    0.31035245103378    0.08285107561837
       0.31035245103378    0.05314504984482    0.08285107561837
       0.05314504984482    0.63650249912140    0.08285107561837];

elseif (n == 7)
   xw=[0.33333333333333    0.33333333333333   -0.14957004446768
       0.26034596607904    0.26034596607904    0.17561525743321
       0.26034596607904    0.47930806784192    0.17561525743321
       0.47930806784192    0.26034596607904    0.17561525743321
       0.06513010290222    0.06513010290222    0.05334723560884
       0.06513010290222    0.86973979419557    0.05334723560884
       0.86973979419557    0.06513010290222    0.05334723560884
```

```
        0.31286549600487    0.63844418856981    0.07711376089026
        0.63844418856981    0.04869031542532    0.07711376089026
        0.04869031542532    0.31286549600487    0.07711376089026
        0.63844418856981    0.31286549600487    0.07711376089026
        0.31286549600487    0.04869031542532    0.07711376089026
        0.04869031542532    0.63844418856981    0.07711376089026];

elseif (n == 8)
   xw=[0.33333333333333    0.33333333333333    0.14431560767779
        0.45929258829272    0.45929258829272    0.09509163426728
        0.45929258829272    0.08141482341455    0.09509163426728
        0.08141482341455    0.45929258829272    0.09509163426728
        0.17056930775176    0.17056930775176    0.10321737053472
        0.17056930775176    0.65886138449648    0.10321737053472
        0.65886138449648    0.17056930775176    0.10321737053472
        0.05054722831703    0.05054722831703    0.03245849762320
        0.05054722831703    0.89890554336594    0.03245849762320
        0.89890554336594    0.05054722831703    0.03245849762320
        0.26311282963464    0.72849239295540    0.02723031417443
        0.72849239295540    0.00839477740996    0.02723031417443
        0.00839477740996    0.26311282963464    0.02723031417443
        0.72849239295540    0.26311282963464    0.02723031417443
        0.26311282963464    0.00839477740996    0.02723031417443
        0.00839477740996    0.72849239295540    0.02723031417443];

        .
        .
        .
        . (cases for n>=9 omitted in the print-out)
        .
        .
        .

else
   error('Bad input n');
end

return
end
```

References

[1] D. A. Dunavant, High degree efficient symmetrical Gaussian quadrature rules for the triangle, Int. J. Num. Meth. Engng, 21(1985):1129-1148.

[2] B. Szabó, I. Babuška, Finite Element Analysis, Wiley, New York, 1991.