Sequences, Series, and Power Series

Example: Consider the sequence \( a_n = \frac{2n+1}{3n} \)

\[
\lim_{n \to \infty} \frac{2n + 1}{3n} = \frac{2}{3}
\]

1. The series formed from this sequence is

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n + 1}{3n} = 1 + \frac{5}{6} + \frac{7}{9} + \ldots
\]

diverges to \( \infty \).

2. The alternating series formed from this sequence is

\[
\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n + 1}{3n} = 1 - \frac{5}{6} + \frac{7}{9} - \ldots
\]

diverges by the alternating series test.

3. A power series formed from this sequence is

\[
\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left( \frac{2n + 1}{3n} \right) x^n = x + \frac{5x^2}{6} + \frac{7x^3}{9} + \ldots
\]

Then the radius of convergence and the interval of convergence are found by determining:

\[
\ell = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2n + 3}{3n + 3} \div \frac{2n + 1}{3n} \right| = \lim_{n \to \infty} \frac{2(n+1) + 3}{3(n+1) + 3} \div \frac{3n + 3}{2n + 1} = 1
\]

by definition, the radius of convergence \( R \) is \( R = \frac{1}{\ell} = 1 \). We know that the power series converges for \(-1 < x < 1\) since

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} x^n \right| = |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| < 1 \quad \text{to converge by the ratio test.}
\]

and diverges on \((-\infty, -1) \cup (1, \infty)\). At \(x = 1\) or \(x = -1\), we must check the resulting series. At \(x = 1\), we get \(\sum_{n=1}^{\infty} \frac{2n+1}{3n}\) which we already saw diverges, and at \(x = -1\), we get \(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{3n}\) which we already saw diverges.

Thus, the interval of convergence is \((-1, 1)\)