Homework Set 23
(sect 8.6: Functions as Power Series)

Find a power series representation for each function. You may want to use the following fact:

\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}, \quad |x| < 1 \]

1. \( f(x) = \frac{1}{1-x^3} = \frac{1}{1-(x^3)} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n} = 1 + x^3 + x^6 + x^9 + \ldots \quad \text{,} \quad |x| < 1 \)

2. \( f(x) = \frac{1}{x^2+3} = \frac{\frac{1}{3}}{1+(\frac{x^2}{3})} = \frac{\frac{1}{3}}{1-(-\frac{x^2}{3})} \)

\[ = \frac{1}{3} \left[ 1 - \frac{x^2}{3} + \frac{x^4}{9} - \frac{x^6}{27} + \ldots \right] = \frac{1}{3} \sum_{n=0}^{\infty} (-\frac{x^2}{3})^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{3^{2n+1}} \quad \text{,} \quad |x| < 3 \quad \text{(w/c \(-\frac{x^2}{3} \leq 1\))} \)

3. \( f(x) = \frac{x}{5x-1} = -x \left[ \frac{1}{1-\frac{x}{5}} \right] = -x \sum_{n=0}^{\infty} \left( \frac{x}{5} \right)^n \)

\[ = -\sum_{n=0}^{\infty} \frac{x^2}{5^n} \quad \text{,} \quad \left| x \right| < \frac{1}{5} \quad \text{(w/c \left| x \right| < 1)} \)

4. \( f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) \quad \text{[Hint: use differentiation.]} \)

\[ = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{d}{dx} \left[ 1 + x + x^2 + x^3 + \ldots \right] \]

\[ = 1 + 2x + 3x^2 + 4x^3 + \ldots \]

\[ = \sum_{n=0}^{\infty} (n+1)x^n \quad \text{,} \quad \left| x \right| < 1 \]

5. \( f(x) = \frac{x^2}{(1-2x)^2} \)

\[ = x^2 \cdot \frac{1}{[1-(2x)]^2} = x^2 \left[ 1 + 2(2x) + 3(2x)^2 + 4(2x)^3 + \ldots \right] \]

\[ = x^2 \sum_{n=0}^{\infty} (n+1)(2x)^n = \sum_{n=0}^{\infty} (2^n)(n+1)x^{n+2} \quad \text{,} \quad \left| x \right| < \frac{1}{2} \]