Summary of Convergence Tests

NAME	STATEMENT	COMMENTS
Divergence Test	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may or may not converge.
P-Test	Let $\sum \frac{1}{n^p}$ be a series with positive terms, then (a) Series converges if $p>1$ (b) Series diverges if $p\leq 1$	This test can be used in conjunction with the comparison test for any a_n whose denominator is raised to the n^{th} power.
Comparison Test	Let $\sum a_n$ and $\sum b_n$ be series with non-negative terms such that $a_1\leq b_1$, $a_2\leq b_2$, $a_3\leq b_3$, … if $\sum b_n$ converges, then $\sum a_n$ converges, and if $\sum a_n$ diverges, then $\sum b_n$ diverges.	Try this test as a last resort since other tests are often easier to apply.
Limit Comparison Test	Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $\ell = \lim_{n\to\infty} \frac{a_n}{b_n}$ if $0 < \ell < \infty$, then both series converge, or both series diverge.	This is easier to apply than the Comparison Test, but requires some intuition in choosing the appropriate series $\sum b_n$ for comparison.
Integral Test	Let $\sum a_n$ be a series with positive terms, and let f(x) be the function that results when n is replaced by x in the n^{th} term of the associated sequence. If f(x) is decreasing and continuous for $x \geq 1$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ both converge or both diverge.	This test only works for series that have positive terms. Try this test when f(x) is easy to integrate.
Ratio Test	Let $\sum a_n$ be a series with positive terms and suppose $\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ (a) Series converges if $\ell < 1$ (b) Series diverges if $\ell > 1$ (c) Test fails if $\ell = 1$	Try this test when a_n involves factorials or n^{th} powers.
Ratio Test for Absolute Convergence	Let $\sum a_n$ be a series and suppose $\ell = \lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\|$ (a) Series converges if $\ell < 1$ (b) Series diverges if $\ell > 1$ (c) Test fails if $\ell = 1$	This is the default test because it is one of the easiest tests and it rarely fails. Note: the series need not have only positive terms nor does it have to be alternating.
Root Test	Let $\sum a_n$ be a series and suppose $\ell = \lim_{n\to\infty} \sqrt[n]{\|a_n\|}$ (a) Series converges if $\ell < 1$ (b) Series diverges if $\ell > 1$ (c) Test fails if $\ell = 1$	This test is the most accurate, but not the easiest to use in many situations. Use this test when a_n has n^{th} powers.
Alternating Series Test	If $a_n > 0$ for all n, then the series $a_1 - a_2 + a_3 - a_4 + \ldots$ or $-a_1 + a_2 - a_3 + a_4 - \ldots$ Converge if the following conditions hold: (a) $a_1 > a_2 > a_3 > a_4 > \ldots$ (b) $\lim_{n\to\infty} a_n = 0$	This test only applies to alternating series.

Summary of Tests for Series

Test	Series	Converges	Diverges	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Strategies for Testing Series

You have now studied ten tests for determining the convergence or divergence of an infinite series. (See the summary in the table on page 593.) Skill in choosing and applying the various tests will come only with practice. Below is a useful checklist for choosing an appropriate test.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

EXAMPLE 5 Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ **d.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$

e. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$ **g.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Solution

a. For this series, the limit of the nth term is not 0 $\left(a_n \to \frac{1}{3} \text{ as } n \to \infty\right)$. Thus, by the nth-Term Test, the series diverges.

b. This series is geometric. Moreover, because the common ratio of the terms is less than 1 in absolute value $(r = \pi/6)$, you can conclude that the series converges.

c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The nth term of this series can be compared to the nth term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose nth term approaches 0. Because $a_{n+1} \le a_n$, you can use the Alternating Series Test to conclude that the series converges.

f. The nth term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The nth term of this series involves a variable that is raised to the nth power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges.