**Convergence of Sequences, Series, & Power Series**

**Example:** Consider the sequence \( a_n = \frac{2n+1}{3n} \)

\[
\lim_{n \to \infty} \frac{2n+1}{3n} = \frac{2}{3} \quad \text{Thus, we see that the sequence converges to } \frac{2}{3}.
\]

1. The series formed from this sequence is

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+1}{3n} = 1 + \frac{5}{6} + \frac{7}{9} + ... 
\]

And it diverges to \( \infty \) by the divergence test since \( a_n \to \frac{2}{3} \neq 0 \).

2. An alternating series formed from this sequence is

\[
\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{3n} = -1 + \frac{5}{6} - \frac{7}{9} + ... 
\]

And it diverges by the alternating series test since \( a_n \to \frac{2}{3} \neq 0 \).

3. A power series formed from this sequence is

\[
\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left( \frac{2n+1}{3n} \right) x^n = x + \frac{5x^2}{6} + \frac{7x^3}{9} + ... 
\]

Then the radius of convergence and the interval of convergence are found by determining:

\[
\ell = \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \lim_{n \to \infty} \left| \frac{\frac{2n+3}{3n+3}}{\frac{2n+1}{3n}} \right| = \lim_{n \to \infty} \left( \frac{2n+3}{3n+3} \right) \left( \frac{3n}{2n+1} \right) = 1
\]

by definition, the radius of convergence \( R \) is \( R = \frac{1}{\ell} - 1 \). So, we know that the power series converges for \(-1 < x < 1\) and diverges on \((\infty, -1) \cup (1, \infty)\). But at \( x = 1 \) and \( x = -1 \), we must check the resulting two series: At \( x = 1 \), we get \( \sum_{n=1}^{\infty} \frac{2n+1}{3n} \) which we already saw diverges, and at \( x = -1 \), we get \( \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{3n} \) which we already saw diverges.

Thus, the interval of convergence is \((-1, 1)\)
Power Series

Find the interval and radius of convergence for each series.

\[ \sum_{n=1}^{\infty} \frac{x^n}{n} \]

\[ \sum_{n=1}^{\infty} n! \cdot x^n \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^n (x - 1)^n}{5^n} \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!} \]

\[ \sum_{n=1}^{\infty} \frac{5(x + 3)^n}{2^n} \]