Leonardo Fibonacci (also known as Leonardo of Pisa) lived during the late 1100’s and early 1200’s. Due to his interest in rabbits, we have the Fibonacci sequence. The terms of this named sequence are referenced by $F_n$ (not the standard $a_n$). The first several terms are listed below:

| $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{13}$ | $F_{14}$ | $F_{15}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0     | 1     | 1     | 2     | 3     | 5     | 8     | 13    | 21    | 34    | 55    | 89    | 144   | 233   | 377   | 610   |

This sequence has a very nice and easy recursive form for $F_n$. To get the next term in the sequence add the previous two terms together: $F_{n+1} = F_n + F_{n-1}$ However, the closed form for $F_n$ is much more complicated: 

$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^{n+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$

Fibonacci-type (or Fibonacci-like) sequences are sequences whose terms are found by adding the previous two terms together. For example: -3, 5, 2, 7, 9, 16, ... They are also sequences whose terms are found by adding multiples of the previous two terms together. For example: 2, 1, 4, 9, 22, 53, ... Here we have $a_{n+1} = 2a_n + a_{n-1}$

In general, the recursive form for $a_n$ is $a_{n+1} = a \cdot a_n + b \cdot a_{n-1}$ (where $a$ and $b$ are constant numbers). But often we don’t want the recursive form. Normally, we need to have the closed form for $a_n$. The nice thing about Fibonacci-type sequences is we can easily find the closed form of $a_n$ from its recursive form.

See the following websites for more information about Fibonacci sequences:

- The Fibonacci sequence found in nature with good photographs: [http://www.environmentalgraffiti.com/featured/fibonacci-sequence-illustrated-nature/10867](http://www.environmentalgraffiti.com/featured/fibonacci-sequence-illustrated-nature/10867)
- More information on Leonardo Fibonacci: [http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fibonacci.html](http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fibonacci.html)
- A basic definition of the Fibonacci sequence and how it is used: [http://www.mathacademy.com/pr/prime/articles/fibonacci/index.asp](http://www.mathacademy.com/pr/prime/articles/fibonacci/index.asp)
- A technical mathematics entry on the Fibonacci sequence and related topics: [http://mathworld.wolfram.com/FibonacciNumber.html](http://mathworld.wolfram.com/FibonacciNumber.html)
Finding the closed form of $a_n$:

Step 1: get the recursive form $a_{n+1} = a \cdot a_n + b \cdot a_{n-1}$ and the first two terms

Step 2: let $\lambda^n = a_n$ so then $\lambda^{n+1} = a_{n+1}$ and $\lambda^{n-1} = a_{n-1}$

Step 3: get the equation $\lambda^{n+1} = a \cdot \lambda^n + b \cdot \lambda^{n-1}$

Step 4: simplify the equation to $\lambda^2 = a \cdot \lambda + b$

Step 5: solve the quadratic equation you got in step 4 by using either factoring or the quadratic equation $if \ ax^2 + bx + c = 0 \ then \ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Step 6: take the two solutions you got from step 5, call them $\lambda_1$ & $\lambda_2$, and set $a_n = c_1 \cdot \lambda_1^n + c_2 \cdot \lambda_2^n$ where $c_1$ & $c_2$ are constants that we don’t know yet

Step 7: solve for $c_1$ & $c_2$ by using the initial two terms of the sequence

Example:

Step 1: $a_{n+1} = 5 \cdot a_n - \frac{9}{4} \cdot a_{n-1}$ and $a_0 = 1 \ , a_1 = 2$

Step 2&3: $\lambda^{n+1} = 5 \cdot \lambda^n - \frac{9}{4} \cdot \lambda^{n-1}$

Step 4: $\lambda^2 = 5 \cdot \lambda - \frac{9}{4}$

Step 5: $\lambda^2 - 5 \cdot \lambda + \frac{9}{4} = 0$

$$\lambda = \frac{5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot (9/4)}}{2} = \frac{5 \pm \sqrt{16}}{2} = \frac{5 \pm 4}{2} = \frac{9}{2} \pm \frac{1}{2}$$

Step 6: $\lambda_1 = \frac{9}{2} \& \lambda_2 = \frac{1}{2}$ so $a_n = c_1 \cdot \left(\frac{9}{2}\right)^n + c_2 \cdot \left(\frac{1}{2}\right)^n$

Step 7: $a_0 = 1$ so we get that $a_0 = 1 = c_1 \cdot \left(\frac{9}{2}\right)^0 + c_2 \cdot \left(\frac{1}{2}\right)^0 \Rightarrow 1 = c_1 + c_2$

$a_1 = 2$ so we get that $a_1 = 2 = c_1 \cdot \left(\frac{9}{2}\right)^1 + c_2 \cdot \left(\frac{1}{2}\right)^1 \Rightarrow 2 = \frac{(9c_1+c_2)}{2} \Rightarrow 4 = 9c_1 + c_2$

So we have the system of equations: $1 = c_1 + c_2 \& 4 = 9c_1 + c_2$

Using algebra we get that $c_1 = \frac{3}{8} \& c_2 = \frac{5}{8}$

So $a_n = \frac{3}{8} \cdot \left(\frac{9}{2}\right)^n + \frac{5}{8} \cdot \left(\frac{1}{2}\right)^n = \frac{3}{2^3} \cdot \left(\frac{3^2}{2}\right)^n + \frac{5}{2^3} \cdot \left(\frac{1}{2}\right)^n = \frac{3^{2n+1}+5}{2^{n+3}}$