Random string: an explicitly solvable model

Yuri A. GODIN and Stanislav MOLCHANOV

Department of Mathematics and Statistics
University of North Carolina at Charlotte
Charlotte, NC 28223

Received for ROSE May 16, 2004

Abstract—We calculate exactly the Lyapunov exponent and the integrated density of states for the random string operator whose density and elastic compliance are functions of a binary Markov chain. For all values of interest short and long wave asymptotic expansions are obtained. Finally we discuss conditions of propagation and localization of waves in a binary random medium.

1. INTRODUCTION

The explicitly solvable model of one-dimensional Schrödinger operator

$$H\psi = -\psi'' + V(t, \omega)\psi, \quad x \in \mathbb{R}^1$$  \hspace{1cm} (1.1)

with Markov type potential $V(t, \omega) = q(x_t)$ was presented in the paper [1]. Here $x_t$ is a Markov chain with two possible states $x = (0, 1)$ (i.e. $V$ has two alternating values $q_0$ and $q_1$). Explicit solvability in this case means that one can find in quadratures the joint limiting distribution density for the pair $(x_t, \vartheta_E(t))$, where $\vartheta_E(t)$ is the phase of the problem $H\psi = E\psi$, i.e. solution of the Riccatti equation

$$\dot{\vartheta}_E(t) = \cos^2 \vartheta_E(t) + (E - q(x_t)) \sin^2 \vartheta_E(t), \quad \vartheta \in [0, \pi) = S^1.$$  \hspace{1cm} (1.2)

As it is easy to see, $(x_t, \vartheta_E(t))$ is an ergodic Markov process on the product space $X \times S^1$ and $p(x, \vartheta)$ is the invariant measure for $(x_t, \vartheta_E(t))$.

In this paper, the integrated density of states $N(E)$ and the Lyapunov exponent will be calculated in closed form in terms of $p(x, \vartheta)$ similar to the corresponding results in [1]-[2]. The goal of this publication is the analysis of the random string operator

$$H\psi = -\frac{1}{\varrho(t)} \frac{d}{dt} \left( \frac{1}{s(t)} \frac{d\psi}{dt} \right),$$  \hspace{1cm} (1.3)

where the density $\varrho(t)$ and the elastic compliance $s(t)$ are the functions of the Markov chain $x_t$, $t \geq 0$, with two values $\{0, 1\} = X$. The case when $x_t$ is a diffusion process has been substantially developed in the literature (see [3]-[5]).

Assume that

$$A = \begin{bmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix}$$  \hspace{1cm} (1.4)
is the matrix of transition intensities for $x_t$, i.e.

$$
\begin{align*}
P\{x_{t+dt} = 0 \mid x_t = 0\} &= 1 - \lambda_0 dt, \\
P\{x_{t+dt} = 0 \mid x_t = 1\} &= \lambda_1 dt, \\
P\{x_{t+dt} = 1 \mid x_t = 0\} &= \lambda_0 dt, \\
P\{x_{t+dt} = 1 \mid x_t = 1\} &= 1 - \lambda_1 dt,
\end{align*}
$$

and invariant probabilities for $x$ are

$$
\begin{align*}
\mu_0 &= \lim_{t \to \infty} P\{x_t = 0\} = \frac{\lambda_1}{\lambda_0 + \lambda_1}, \\
\mu_1 &= \lim_{t \to \infty} P\{x_t = 1\} = \frac{\lambda_0}{\lambda_0 + \lambda_1}.
\end{align*}
$$

The string consists of two alternating components of random length (type ‘0’ and type ‘1’) with density $\rho_i$ and elastic stiffness $s_i$, $i = 0, 1$, respectively. The length of components of each type $\ell_i$ is distributed exponentially with parameters $\lambda_i$:

$$
P\{\ell_i > x\} = e^{-\lambda_i x}, \quad i = 0, 1.
$$

For the mean values of $\rho$ and $s$ over ensemble of realizations we have

$$
\langle s \rangle = \frac{s_0 \lambda_1 + s_1 \lambda_0}{\lambda_0 + \lambda_1}, \quad \langle \rho \rangle = \frac{\rho_0 \lambda_1 + \rho_1 \lambda_0}{\lambda_0 + \lambda_1}.
$$

The spectrum of $H$ coincides with $\sum(H) = [0, \infty)$. For $E = \omega^2 > 0$ ($E = -\omega^2$ when $E < 0$) we introduce the phase-amplitude representation of the equation $H \psi = \omega^2 \psi$

$$
\begin{align*}
\psi(t) &= r_E(t) \sin \vartheta_E(t), & r_E &= \sqrt{\psi^2 + \left(\frac{\psi'}{s}\right)^2}, \\
\frac{\psi'(t)}{s(t)} &= r_E(t) \cos \vartheta_E(t), & \vartheta_E \in S^1 = [0, \pi).
\end{align*}
$$

The phase and the amplitude then satisfy the system

$$
\begin{align*}
\frac{d\vartheta_E}{dt} &= s(t) \cos^2 \vartheta_E(t) + E \rho(t) \sin^2 \vartheta_E, \\
\frac{dr_E}{dt} &= \frac{1}{2} r_E \sin 2\vartheta_E [s(t) - E \rho(t)].
\end{align*}
$$
The generator of the Markov process \((x_t, \vartheta_E(t))\) acts on the vector functions \(f(x, \vartheta) = [f_0(\vartheta), f_1(\vartheta)]\) and is given by
\[
\mathcal{L}f = \begin{bmatrix}
-\lambda_0 & \lambda_0 \\
\lambda_1 & -\lambda_1
\end{bmatrix} f + \begin{bmatrix}
a_0(\vartheta) \frac{df_0}{dt} \\
a_1(\vartheta) \frac{df_1}{dt}
\end{bmatrix},
\]
(1.10)
where
\[
a_i(\vartheta) = s_i \cos^2 \vartheta + E \vartheta_i \sin^2 \vartheta, \quad i = 0, 1.
\]
(1.11)
For the invariant density \(p(x, \vartheta) = [p_0(\vartheta), p_1(\vartheta)]^T\) one has \(\mathcal{L}^*p = 0\), or
\[
\mathcal{L}^*p = \begin{bmatrix}
-\lambda_0 & \lambda_1 \\
\lambda_0 & -\lambda_1
\end{bmatrix} p(\vartheta) - \begin{bmatrix}
\frac{\partial}{\partial \vartheta} (a_0(\vartheta)p_0(\vartheta)) \\
\frac{\partial}{\partial \vartheta} (a_1(\vartheta)p_1(\vartheta))
\end{bmatrix} = 0.
\]
(1.12)
Thus, \(p(\vartheta)\) satisfies the linear system
\[
(a_0(\vartheta)p_0(\vartheta))' = -\lambda_0 p_0 + \lambda_1 p_1,
\]
(1.13)
\[
(a_1(\vartheta)p_1(\vartheta))' = \lambda_0 p_0 - \lambda_1 p_1,
\]
(1.14)
whose first integral is
\[
a_0(\vartheta)p_0(\vartheta) + a_1(\vartheta)p_1(\vartheta) = c.
\]
(1.15)
Calculation of \(p(x, \vartheta)\) is given in §2. Below we present basic formulas for the integrated density of states \(N(\vartheta_E), \quad E \geq 0\), and the Lyapunov exponent \(\gamma(E), \quad E \in \mathbb{R}\) in terms of \(p(\cdot)\). In [2] on can find similar formulas for \(N(\vartheta_E)\) (in the case of 1D Schrödinger operator) but with a minor error.

1.1. Integrated density of states
From Sturm’s oscillation theorem and integration by parts we obtain
\[
N(\vartheta_E) = \lim_{L \to \infty} \frac{1}{\pi L} \int_0^L d\vartheta_E(u) - \frac{1}{\pi} \langle s(x) \cos^2 \vartheta_E(\cdot) + E \vartheta_E(x) \sin^2 \vartheta_E(\cdot) \rangle
\]
\[
= \frac{1}{\pi} \int_0^\pi \left[ a_0(\vartheta)p_0(\vartheta) + a_1(\vartheta)p_1(\vartheta) \right] d\vartheta
\]
\[
= \frac{1}{\pi} \left\{ \left[\vartheta a_0(\vartheta)p_0(\vartheta) + \vartheta a_1(\vartheta)p_1(\vartheta) \right]_0^\pi - \int_0^\pi \vartheta \left[ a_0(\vartheta)p_0(\vartheta) + a_1(\vartheta)p_1(\vartheta) \right] d\vartheta \right\},
\]
(1.15)
where the integrand vanishes by (1.14). Therefore,
\[
N(\vartheta_E) = s_0 p_0(0) + s_1 p_1(0).
\]
(1.16)
Comparison of (1.16) and (1.14) yields that the meaning of constant \(c\) is the value of the integrated density of states for given frequency \(\omega\) or energy \(E\)
\[
c = N(\vartheta_E).
\]
(1.17)
1.2. Lyapunov exponent

The Lyapunov exponent determines asymptotic decay of the solution. By definition,

\[
\gamma(E) = \lim_{L \to \infty} \left( \frac{1}{L} \right) \ln r_E(L)
\]

\[
= \frac{1}{L} \int_0^L \frac{1}{2} \sin 2 \vartheta_E(s(x_u) - E \varrho(x_u)) \, du
\]

\[
= \frac{1}{2} \int_0^\pi \sin 2 \vartheta(s_0 - E \varrho_0) p_0(\vartheta) \, d\vartheta
\]

\[
+ \frac{1}{2} \int_0^\pi \sin 2 \vartheta(s_1 - E \varrho_1) p_1(\vartheta) \, d\vartheta.
\]

We will present explicit asymptotic formulas for \( N(E) \) when \( E \to 0 \) or \( E \to +\infty \), and for \( \gamma(E) \) when \( E \to 0 \) or \( E \to \pm \infty \). Comparison of the corresponding formulas in the Schrödinger case and calculation of \( \gamma(E) \) will be given elsewhere (for calculation of \( N(E) \) see [1]-[2]). Of particular interest here is the formula for \( N(E) \approx c_0 \sqrt{E} \), \( \gamma(0) = 0 \), which expresses the phenomenon of localization for the wave (or heat) equations associated with Hamiltonian \( H \) (see §3), and the relationship between localization phenomenon for operator \( H \) on the positive part of the spectrum of \( H \), and homogenization corresponding to the degeneracy of \( \gamma(E) \) in the limit of very long waves (cf. with discussion in [5]).

2. INVARIANT DENSITIES \( p_i(\vartheta) \), \( i = 0,1 \)

Solution of the system (1.13) gives the first integral (1.14). Taking into account periodicity

\[
p_i(0) = p_i(\pi), \quad i = 0,1,
\]

and normalization conditions

\[
\int_0^\pi p_i(\vartheta) \, d\vartheta = \frac{\lambda_1 - i}{\lambda_0 + \lambda_1}, \quad i = 0,1,
\]

we find the invariant densities

\[
p_i(\vartheta) = \begin{cases} \frac{c \lambda_1 - i}{a_i(\vartheta)} \left\{ K_i e^{-\int_0^\vartheta \eta(\vartheta') \, d\vartheta'} \right\} = \frac{\lambda_0}{a_0(\vartheta)} + \frac{\lambda_1}{a_1(\vartheta)}, & i = 0,1, \end{cases}
\]

where

\[
\eta(\vartheta) = \frac{\lambda_0}{a_0(\vartheta)} + \frac{\lambda_1}{a_1(\vartheta)},
\]

\[
K_i = \int_0^\pi \frac{d\vartheta'}{a_1-i(\vartheta')} e^{-\int_{\vartheta'}^{\vartheta} \eta(\vartheta'') \, d\vartheta''} \frac{1}{1 - e^{-\int_0^\pi \eta(\vartheta') \, d\vartheta'}}, \quad i = 0,1,
\]
\[c^{-1} = \sum_{i=0}^{1} \lambda_{1-i} \int_{0}^{\pi} \left[ K_i e^{-\int_{0}^{\theta} \eta(\theta') \, d\theta'} \right. \]
\[+ \int_{0}^{\theta} \frac{d\theta'}{\alpha_{1-i}(\theta')} e^{-\int_{\phi}^{\theta} \eta(\phi') \, d\phi'} \right] d\theta \]  
\[(2.6)\]

3. SERIES EXPANSION OF THE INTEGRALS ABOUT \(\omega = 0\)

Long wave approximation of \(N(E)\) and \(\gamma(E)\) requires calculation of asymptotics of integrals involved in (2.3)-(2.6). Using the values of integrals given in Appendix we obtain

\[K_i = \int_{0}^{\pi} \frac{d\theta}{\alpha_{1-i}(\theta)} e^{-\int_{0}^{\theta} \eta(\theta') \, d\theta'} \]
\[\approx \int_{\pi/2}^{\pi} \frac{d\theta}{s_{1-i} \cos^2 \theta + \omega^2 \varphi_{1-i} \sin^2 \theta} e^{-\int_{\varphi}^{\theta} \eta(\varphi') \, d\varphi'} \]
\[\approx \int_{0}^{\pi/2} \frac{d\alpha}{s_{1-i} \varphi_{1-i} \cos^2 \alpha} e^{-\int_{0}^{\pi/2} \left( \frac{\lambda_0}{s_0 \sin^2 \beta} + \frac{\lambda_1}{s_1 \sin^2 \beta} \right) d\beta} \]
\[= \frac{s_i}{\lambda_0 s_1 + \lambda_1 s_0}, \quad i = 0, 1, \quad \omega \to 0. \]  
\[(3.1)\]

Let us find now the asymptotics of \(c\). Evaluation of the first and the third terms in (2.6) gives

\[\int_{0}^{\pi} \frac{d\theta}{\alpha_i(\theta)} e^{-\int_{0}^{\theta} \eta(\theta') \, d\theta'} \sim \int_{0}^{\pi/2} \frac{d\theta}{\alpha_i(\theta)} e^{-\int_{0}^{\theta} \eta(\theta') \, d\theta'} \]
\[\sim \int_{0}^{\pi/2} \frac{d\theta}{s_i \cos^2 \theta} e^{-\int_{0}^{\theta} \left( \frac{\lambda_0}{s_0} + \frac{\lambda_1}{s_1} \right) \cos^2 \theta' \, d\theta'} \]
\[= \frac{s_{1-i}}{\lambda_0 s_1 + \lambda_1 s_0}, \quad i = 0, 1, \quad \omega \to 0. \]  
\[(3.2)\]

Asymptotics of the second and fourth term in (2.6) has different form. Take for example the second term in (2.6)

\[\int_{0}^{\pi} \frac{d\theta}{\alpha_i(\theta)} \int_{0}^{\theta} \frac{d\theta'}{\alpha_{1-i}(\theta')} e^{-\int_{0}^{\phi} \eta(\phi') \, d\phi'} \sim \int_{0}^{\pi} \frac{d\theta}{\langle s \rangle \cos^2 \theta} e^{-\int_{0}^{\theta} \left( \frac{\lambda_0 + \lambda_1}{\varphi} \right)^{-1} \cos^2 \phi \, d\phi} \]
\[= \frac{\pi}{\omega(s_{0} + s_{1}) \sqrt{s_{0}s_{1}}}, \quad i = 0, 1. \]  
\[(3.3)\]
From the above formulas we find the asymptotics of \( N(\omega) \)

\[
N(\omega) = c \sim \frac{\omega}{\pi} \sqrt{\langle s \rangle \langle \rho \rangle}, \quad \omega \to 0.
\]

Calculation of the integrals in (1.18) gives the asymptotic value of the Lyapunov exponent

\[
\gamma(\omega) \sim \frac{\omega^2}{4} \frac{\lambda_0 \lambda_1}{(\lambda_0 + \lambda_1)^2} \frac{(s_0 \theta_1 - s_1 \theta_0)^2}{\langle \theta \rangle \langle s \rangle}, \quad \omega \to 0.
\]

(3.4)

(3.5)

4. SERIES EXPANSION OF THE INTEGRALS ABOUT \( \omega = \infty \)

Below we calculate short wave asymptotics of integrals using formulas in Appendix.

\[
K_i = \int_0^\pi \frac{d\theta}{a_{1-i}(\theta)} e^{-\int_0^\pi \frac{\lambda_0}{\sqrt{s_0 \theta_0}} + \frac{\lambda_1}{\sqrt{s_1 \theta_1}}} \left( -\frac{\lambda_0}{\sqrt{s_0 \theta_0}} + \frac{\lambda_1}{\sqrt{s_1 \theta_1}} \right) \left( \lambda_0 \lambda_1 \omega \right)
\]

\[
\sim 2 \int_0^{\pi/2} \frac{d\theta}{s_{1-i} \cos^2 \theta + \omega^2 \theta_{1-i} \sin^2 \theta} \left( \frac{\lambda_0}{\sqrt{s_0 \theta_0}} + \frac{\lambda_1}{\sqrt{s_1 \theta_1}} \right) \left( \lambda_0 \lambda_1 \omega \right)
\]

\[
\sim \frac{\pi}{\omega} \left( \frac{\lambda_0}{\sqrt{s_0 \theta_0}} + \frac{\lambda_1}{\sqrt{s_1 \theta_1}} \right) \left( \lambda_0 \lambda_1 \omega \right), \quad i = 0, 1, \omega \to \infty.
\]

(4.1)

The main contributors to asymptotics of \( c \) are the first and second term in (2.6). Therefore, from (4.1) and (A.1), (A.3) it follows that

\[
c \sim \frac{\omega \lambda_0 \sqrt{s_1 \theta_1} + \lambda_1 \sqrt{s_0 \theta_0}}{\lambda_0 + \lambda_1} \quad \text{as} \quad \omega \to \infty.
\]

(4.2)

Expression for the Lyapunov exponent (1.18) can also be written in the following form

\[
\gamma(\omega) = \frac{1}{2} (\omega^2 \theta_0 - s_0) \int_0^{\pi/2} [p_0(\pi - \theta) - p_0(\theta)] \sin 2\theta \, d\theta
\]

\[
+ \frac{1}{2} (\omega^2 \theta_1 - s_1) \int_0^{\pi/2} [p_1(\pi - \theta) - p_1(\theta)] \sin 2\theta \, d\theta.
\]

(4.3)

Calculating the integrands in (4.3) using (2.3), we obtain

\[
\gamma(\omega) \sim \frac{c \lambda_0 \lambda_1 \omega}{\lambda_0 \sqrt{s_1 \theta_1} + \lambda_1 \sqrt{s_0 \theta_0}}
\]
Random string: an explicitly solvable model

\[
\times \left\{ \frac{\varrho_0}{s_0} \int_0^{\pi/2} \frac{\arctan(\sqrt{\varrho_0 \tan \vartheta})}{s_0 \cos^2 \vartheta + \omega^2 \varrho_0 \sin^2 \vartheta} \sin 2\vartheta \, d\vartheta - \frac{\varrho_1}{s_1} \int_0^{\pi/2} \frac{\arctan(\sqrt{\varrho_1 \tan \vartheta})}{s_1 \cos^2 \vartheta + \omega^2 \varrho_1 \sin^2 \vartheta} \sin 2\vartheta \, d\vartheta \right\}
\]

\[
+ \varrho_1 \int_0^{\pi/2} \frac{\arctan(\sqrt{\varrho_1 \tan \vartheta})}{s_1 \cos^2 \vartheta + \omega^2 \varrho_1 \sin^2 \vartheta} \sin 2\vartheta \, d\vartheta \right) \right\}
\]

\[
\lesssim \frac{2\lambda_0 \lambda_1}{\pi (\lambda_0 + \lambda_1)} \left( \frac{s_1}{\varrho_1} - \frac{s_0}{\varrho_0} \right) \left( \frac{\varrho_0}{s_0} - \frac{\varrho_1}{s_1} \right) \int_0^{\pi/2} \omega^3 \tan^2 \vartheta \left( \frac{s_0 + \omega^2 \tan^2 \vartheta}{\varrho_0} \right) \left( \frac{s_1 + \omega^2 \tan^2 \vartheta}{\varrho_1} \right) \left( 1 + \omega^2 \sqrt{\frac{\varrho_0 \varrho_1}{s_0 s_1}} \tan^2 \vartheta \right) \, d\vartheta.
\]

Evaluation of the latter integral gives

\[
\gamma(\omega) \sim \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \left( \frac{s_1 \varrho_0}{s_0 \varrho_1} \right)^{1/8} - \left( \frac{s_0 \varrho_1}{s_1 \varrho_0} \right)^{1/8} \right]^2, \quad \omega \to \infty.
\]

5. ASYMPTOTICS OF THE LYAPUNOV EXPONENT IN THE CASE OF NEGATIVE ENERGY

When the energy is negative, \( E = -\omega^2 \), it follows from (1.9) that invariant density does not vanish only on the interval \((\vartheta_0, \vartheta_1)\), where

\[
\cot \vartheta_0 = \omega \sqrt{\frac{\varrho_0}{s_0}} \quad \text{and} \quad \cot \vartheta_1 = \omega \sqrt{\frac{\varrho_1}{s_1}}.
\]

(5.1)

We will suppose that

\[
\frac{\varrho_0}{s_0} > \frac{\varrho_1}{s_1} \quad \text{so that} \quad \vartheta_0 \leq \vartheta_1.
\]

(5.2)

Then solution of (1.13) has a simple form

\[
p_i(\vartheta) = \frac{C_i}{a_i(\vartheta)} e^{-\int_{\vartheta_*}^{\vartheta} \eta(t) \, dt}, \quad i = 0, 1,
\]

(5.3)

where now

\[
a_i(\vartheta) = s_i \cos^2(\vartheta) - \omega^2 \sin^2(\vartheta), \quad i = 0, 1,
\]

(5.4)

and the lower limit of integration \( \vartheta_* \) is chosen in such a way that

\[
\eta(\vartheta_*) = 0 \quad \text{or} \quad \vartheta_* = \cot^{-1} \omega \sqrt{\frac{(\vartheta)^2}{(s)}},
\]

(5.5)

Constants \( C_i, \ i = 0, 1, \) can be found from the normalization condition

\[
C_i \int_{\vartheta_0}^{\vartheta} \frac{1}{a_i(\vartheta)} e^{-\int_{\vartheta_*}^{\vartheta} \eta(t) \, dt} \, d\vartheta = \frac{\lambda_i}{\lambda_0 + \lambda_1}, \quad i = 0, 1.
\]

(5.6)
Similar to (1.18), the Lyapunov exponent has the form
\[
\gamma(E) = \frac{1}{2}(s_0 + E\rho_0) \int_{\vartheta_0}^{\vartheta_1} p_0(\vartheta) \sin 2\vartheta \, d\vartheta + \frac{1}{2}(s_1 + E\rho_1) \int_{\vartheta_0}^{\vartheta_1} p_1(\vartheta) \sin 2\vartheta \, d\vartheta. \quad (5.7)
\]

Consider first the case when \( E = -\omega^2 \rightarrow 0 \).

\[
\gamma(E) \sim \frac{1}{2} \int_{\vartheta_0}^{\vartheta_1} \left[ s_0 p_0(\vartheta) + s_1 p_1(\vartheta) \right] \sin 2\vartheta \, d\vartheta \\
= \frac{1}{2} \int_{\frac{\pi}{2} - \sqrt{\pi/2}}^{\frac{\pi}{2} + \sqrt{\pi/2}} \left[ s_0 p_0(\vartheta) + s_1 p_1(\vartheta) \right] \sin 2\vartheta \, d\vartheta \\
= \frac{1}{2} \sqrt{E} \int_{\sqrt{\pi/2}}^{\sqrt{\pi/2}} \left[ s_0 p_0 \left( \frac{\pi}{2} - \sqrt{E\alpha} \right) + s_1 p_1 \left( \frac{\pi}{2} - \sqrt{E\alpha} \right) \right] \sin 2\sqrt{E\alpha} \, d\alpha
\]

\[
\sim \frac{\lambda_1 s_0 \omega}{\lambda_0 + \lambda_1} \int_{\sqrt{\pi/2}}^{\sqrt{\pi/2}} e^{-\frac{1}{\omega} \sqrt{\frac{E\alpha^2}{\pi} + s_0 \alpha^2}} \tilde{\eta}(\beta) \, d\beta \\
\times \left[ \int_{\sqrt{\pi/2}}^{\sqrt{\pi/2}} e^{-\frac{1}{\omega} \sqrt{\frac{E\alpha^2}{\pi} + s_0 \alpha^2}} \tilde{\eta}(\beta) \, d\beta \right]^{-1}
\]

\[
= \frac{\lambda_0 s_1 \omega}{\lambda_0 + \lambda_1} \int_{\sqrt{\pi/2}}^{\sqrt{\pi/2}} e^{-\frac{1}{\omega} \sqrt{\frac{E\alpha^2}{\pi} + s_1 \alpha^2}} \tilde{\eta}(\beta) \, d\beta \\
\times \left[ \int_{\sqrt{\pi/2}}^{\sqrt{\pi/2}} e^{-\frac{1}{\omega} \sqrt{\frac{E\alpha^2}{\pi} + s_1 \alpha^2}} \tilde{\eta}(\beta) \, d\beta \right]^{-1}
\]

(5.8)

where

\[
\tilde{\eta}(\beta) = \frac{\lambda_0}{s_0 \beta^2 - \rho_0} + \frac{\lambda_1}{s_1 \beta^2 - \rho_1}. \quad (5.9)
\]

Finally, calculating the asymptotics of the Laplace integrals in (5.8) we obtain

\[
\gamma(\omega) \sim \omega \sqrt{\langle s \rangle/\langle \theta \rangle}, \quad \omega \rightarrow 0_. \quad (5.10)
\]

The case of large negative energy, \( E = -\omega^2 \), \( \omega \rightarrow +\infty \) is similar to the previous one. From (5.1) and (5.7) we have

\[
\gamma(\omega) \sim \frac{1}{2} \omega^2 \rho_0 C_0 \int_{\frac{\pi}{2} - \sqrt{\pi/2}}^{\frac{\pi}{2} + \sqrt{\pi/2}} \sin 2\vartheta \, d\vartheta - \int_{\vartheta_0}^{\vartheta_1} \eta(t) \, dt \\
\sim \frac{1}{2} \omega^2 \rho_0 C_0 \int_{\sqrt{\pi/2}}^{\sqrt{\pi/2}} \sin 2\sqrt{E\alpha} \, d\alpha - \int_{\vartheta_0}^{\vartheta_1} \eta(t) \, dt
\]

(5.10)
\[ + \frac{1}{2} \omega^2 \varrho_1 C_1 \int \frac{\sin 2\vartheta}{a_1(\vartheta)} e^{-\int_{\vartheta_*}^{\vartheta} \eta(t) \, dt} \, d\vartheta \]

\[ \sim \varrho_0 C_0 \int \sqrt{\frac{\lambda_1}{\lambda_0}} \frac{\alpha}{s_0 - \varrho_0 \alpha^2} e^{-\frac{1}{\omega} \int_{\beta_*}^{\beta} \tilde{\eta}(\beta) \, d\beta} \]

\[ + \varrho_1 C_1 \int \sqrt{\frac{\lambda_1}{\lambda_0}} \frac{\alpha}{s_1 - \varrho_1 \alpha^2} e^{-\frac{1}{\omega} \int_{\beta_*}^{\beta} \tilde{\eta}(\beta) \, d\beta} \]

\[ \sim \frac{\omega}{\lambda_0 + \lambda_1} \sum_{i=0}^{i=1} \lambda_{1-i} \varrho_i \left( \int_0^{\frac{\omega}{\alpha}} \frac{1}{x^2} e^{-\frac{1}{\omega} \int_{\beta_*}^{\beta} \tilde{\eta}(\beta) \, d\beta} \right) \]

\[ \times \int_0^{\frac{\omega}{\alpha}} \frac{1}{x^2} e^{-\frac{1}{\omega} \int_{\beta_*}^{\beta} \tilde{\eta}(\beta) \, d\beta} \, dx, \quad (5.11) \]

where

\[ \tilde{\eta}(\beta) = \frac{\lambda_0}{s_0 - \varrho_0 \beta^2} + \frac{\lambda_1}{s_1 - \varrho_1 \beta^2}. \quad (5.12) \]

Since both integrals contain identical exponents, we obtain for the asymptotic value

\[ \gamma(\omega) \sim \frac{\omega}{\lambda_0 + \lambda_1} (\lambda_0 \sqrt{\varrho_1 s_1} + \lambda_1 \sqrt{\varrho_0 s_0}), \quad E = -\omega^2, \quad \omega \to +\infty. \quad (5.13) \]

6. CONCLUSION

Both integrated density of states \( N(\omega) \), \( E = \omega^2 > 0 \), \( E = -\omega^2 < 0 \), and the Lyapunov exponent \( \gamma(\omega) \) studied in the paper have important thermodynamical meaning: the Laplace transform of \( N(E) \)

\[ \tilde{N}(\beta) = \int_0^\infty e^{-\beta E} dN(E) \quad (6.14) \]

gives statistical sum for the system of electrons in binary random medium, while \( \gamma(\omega) \) denotes the free energy in a model of one-dimensional random polymer [4]. Graphs of both quantities as functions of \( \omega \) are presented in figure 1.
Asymptotic behavior of both functions for $\omega \to 0, \omega > 0$, reflects the fact of homogenization of the random string for very long wavelengths:

\[
\gamma(\omega) \sim \frac{\omega^2}{4} \frac{\lambda_0 \lambda_1}{(\lambda_0 + \lambda_1)^2} \frac{(s_0 \rho_1 - s_1 \rho_0)^2}{\langle s \rangle \langle \rho \rangle},
\]

\[
N(\omega) \sim \frac{\omega}{\pi} \sqrt{\langle \rho \rangle \langle s \rangle}.
\]

Indeed, the average constants of the string are $\langle \rho \rangle$ and $\langle s \rangle$, and the corresponding homogenized operator has the form

\[
\mathcal{H} = -\frac{1}{\langle \rho \rangle \langle s \rangle} \frac{d^2}{dx^2}
\]

whose integrated density of states coincides with (6.16), $\omega \geq 0$. Positivity of $\gamma(\omega), \omega > 0$, means that operator $\mathcal{H}$ has pure point spectrum with probability one that implies localization (on the relationship between localization and homogenization see discussion in [5]). Observe also that (3.5) vanishes as $\omega \to 0$ and this is the reason of homogenization. The coefficient of $\omega^2$ in (3.5) depends on parameters $\lambda_0$ and $\lambda_1$ (whose inverse are concentration of ‘0’ and ‘1’ components) and mechanical constants $\rho_i$ and $s_i, i = 0, 1$, and can be made small in two cases:
Both functions $\gamma(\omega)$ and $N(\omega)$ have only one point of non-analyticity: $\omega = 0$ (i.e. the edge of the spectrum). For the random Schrödinger operator the situation is different. Finally, we note that $N_L(E)$ and $\gamma_L(E)$

\begin{align*}
N_L(E) &= \frac{1}{\pi L} \int_0^L \left( s(x_u) \cos^2 \varphi_E(u) + E \varphi(x_u) \sin^2 \varphi_E(u) \right) \, du \\
\gamma_L(E) &= \frac{1}{2L} \int_0^L \sin 2\varphi_E(u)(s(x_u) + E \varphi(x_u)) \, du
\end{align*}

are additive functionals of the ergodic Markov process $\varphi_E(u) \mod \pi, x_u$, $u \leq L$ and as the result, their fluctuations satisfy the central limit theorem. Analysis of the variances of the fluctuations will be the subject of future studies.

**APPENDIX**

Below we give the values of some integrals used in the paper.

\begin{align*}
\int_0^\pi \eta(\vartheta) \, d\vartheta &= \frac{\pi}{\omega} \left( \frac{\lambda_0}{\sqrt{s_0 \varrho_0}} + \frac{\lambda_1}{\sqrt{s_1 \varrho_1}} \right), \quad (A.1)
\end{align*}

where

\begin{align*}
\eta(\vartheta) &= \frac{\lambda_0}{s_0 \cos^2 \vartheta + \omega^2 \varrho_0 \sin^2 \vartheta} + \frac{\lambda_1}{s_1 \cos^2 \vartheta + \omega^2 \varrho_1 \sin^2 \vartheta}.
\int_0^\vartheta \eta(\vartheta') \, d\vartheta'
= \frac{\lambda_0}{\omega \sqrt{s_0 \varrho_0}} \left[ \frac{\pi}{2} - \arctan \frac{\cot \vartheta}{\omega \sqrt{s_0 \varrho_0}} \right] + \frac{\lambda_1}{\omega \sqrt{s_1 \varrho_1}} \left[ \frac{\pi}{2} - \arctan \frac{\cot \vartheta}{\omega \sqrt{s_1 \varrho_1}} \right] \quad (A.2)
\end{align*}
\[ \int_0^\pi \eta(\vartheta') d\vartheta' \]

\[ = \frac{\lambda_0}{\omega \sqrt{\varrho_0 \varrho_0}} \left[ \frac{\pi}{2} + \arctan \frac{\varrho_0}{\omega} \sqrt{\frac{s_0}{\varrho_0}} \right] + \frac{\lambda_1}{\omega \sqrt{\varrho_1 \varrho_1}} \left[ \frac{\pi}{2} + \arctan \frac{\varrho_1}{\omega} \sqrt{\frac{s_1}{\varrho_1}} \right] \quad (A.3) \]

\[ \int_0^\theta \eta(\vartheta'') d\vartheta'' = \frac{\lambda_0}{\omega \sqrt{\varrho_0 \varrho_0}} \left[ \arctan \frac{\varrho_0}{\omega} \sqrt{\frac{s_0}{\varrho_0}} - \arctan \frac{\varrho_0}{\omega} \sqrt{\frac{s_0}{\varrho_0}} \right] \]

\[ + \frac{\lambda_1}{\omega \sqrt{\varrho_1 \varrho_1}} \left[ \arctan \frac{\varrho_1}{\omega} \sqrt{\frac{s_1}{\varrho_1}} - \arctan \frac{\varrho_1}{\omega} \sqrt{\frac{s_1}{\varrho_1}} \right] \quad (A.4) \]

REFERENCES


Copyright of Random Operators & Stochastic Equations is the property of VSP International Science Publishers. The copyright in an individual article may be maintained by the author in certain cases. Content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.