Propagation of longitudinal waves in a random binary rod

YURI A. GODIN∗

Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

(Received 21 August 2005; in final form 13 December 2005)

Propagation of waves in a composite elastic rod consisting of rods with alternating properties of random length is considered. We calculate exactly the Lyapunov exponent and find its short and long wave asymptotics. Finally, we discuss conditions for propagation and localization of waves in a binary random medium.

1. Introduction

The one-dimensional wave equation describes many phenomena in physics [1]. In a periodic medium, the solution of the wave equation has a Floquet–Bloch structure. Very long waves propagate without attenuation and the medium can be viewed as homogeneous. However, for higher frequencies there are certain intervals (so-called gaps) where waves cannot propagate even though the system is perfectly conservative. Due to multiple scattering, destructive interference occurs, so that the wave amplitude decreases exponentially in the medium. The rate of decay of the solution is called the Lyapunov exponent (the inverse of the localization length). The Lyapunov exponent is strictly positive inside the gaps and equals zero in the spectral bands.

The Floquet–Bloch structure of the solution is completely destroyed if the medium is not perfectly periodic because of random independent variations of geometric or material parameters [2]. Now as the wave propagates, its amplitude decreases exponentially with probability one and the medium can be considered homogeneous for long waves only on a finite time interval. This phenomenon is known as localization.

Calculation of the Lyapunov exponent for disordered systems usually employs the transfer matrix formalism, asymptotic analysis, and numerical modelling [3–6]. Our approach is based on the phase-amplitude representation of the solution and stochastic methods. This allows us to find exactly the joint probability density distribution for the phase of the solution that leads to an exact formula for the Lyapunov exponent.

∗E-mail: ygodin@uncc.edu
2. Formulation of the problem

We consider longitudinal vibrations in an infinite elastic rod composed of alternating rods made of two types of materials $M_0$ and $M_1$. The properties of successive rods alternate and take on only two values: either $\varrho_0, s_0$ in $M_0$ or $\varrho_1, s_1$ in $M_1$ (figure 1), where $\varrho$ is the density of the rod’s material and $s$ is its elastic compliance (the inverse of the Young modulus). We also suppose that the random length $\ell_i$ of the $i$-th component is distributed exponentially with parameters $\lambda_i$

\[
P\{\ell_i > x\} = e^{-\lambda_i x}, \quad i = 0, 1.
\]

Assuming harmonic time dependence of the displacement $u(x,t) = X(x)e^{-i\omega t}$ in the $x$-direction, $X(x)$ must satisfy the wave equation

\[
\frac{1}{\varrho(x)} \frac{d}{dx} \left( \frac{1}{s(x)} \frac{dX}{dx} \right) + \omega^2 X = 0
\]

with

\[
\varrho(x) = \begin{cases} 
\varrho_0, & x \in M_0; \\
\varrho_1, & x \in M_1;
\end{cases} \quad \text{and} \quad s(x) = \begin{cases} 
s_0, & x \in M_0; \\
s_1, & x \in M_1.
\end{cases}
\]

To obtain analytical results for the asymptotic behaviour of the solution, we introduce the phase $\vartheta(x)$ and the amplitude $r(x)$ using the Prüfer transform [7]

\[
X(x) = r(x) \sin \vartheta(x), \quad \vartheta \in [0, \pi) = S^1,
\]

\[
X'(x) = s(x)r(x)\cos \vartheta(x).
\]

Then $r(x)$ and $\vartheta(x)$ obey the system

\[
\frac{d\vartheta}{dx} = s(x) \cos^2 \vartheta + \omega^2 \varrho(x) \sin^2 \vartheta,
\]

\[
\frac{dr}{dx} = \frac{1}{2} r \sin 2\vartheta [s(x) - \omega^2 \varrho].
\]

Following [8, 9], it is constructive to view the rod as a Markov chain $\xi_x$ with two states $\xi = \{0, 1\}$ depending on whether $x \in M_0$ or $x \in M_1$. From (1) we derive transition intensities for $\xi_x$

\[
P\{\xi_{x+dx} = 0 \mid \xi_x = 1\} = \lambda_1 dx,
\]

\[
P\{\xi_{x+dx} = 1 \mid \xi_x = 0\} = \lambda_0 dx,
\]

so $(\xi_x, \vartheta(x))$ is an ergodic Markov process on the product space $\xi \times S^1$ and

\[
p(\xi, \vartheta) = \begin{cases} 
p_0(\vartheta), & \xi = 0; \\
p_1(\vartheta), & \xi = 1,
\end{cases}
\]

is the invariant probability measure for $(\xi, \vartheta)$. For example, the probability that the random phase belongs to the interval $(\vartheta, \vartheta + d\vartheta)$ while $x \in M_0$ is $p_0(\vartheta)d\vartheta$. 

Figure 1. An elastic binary rod composed of two alternating materials with parameters $\varrho_0, s_0$ and $\varrho_1, s_1$, where $\varrho$ is the density and $s$ is the elastic compliance, respectively.
The infinitesimal operator \( \mathcal{L} \) of the Markov process \((\xi, \vartheta(x))\) acts on the vector functions \( f(\xi, \vartheta) = [f_0(\vartheta), f_1(\vartheta)] \) and is given by [9]

\[
\mathcal{L} f = \begin{bmatrix}
-\lambda_0 & \lambda_0 \\
\lambda_1 & -\lambda_1
\end{bmatrix} f + \begin{bmatrix}
a_0(\vartheta) \frac{df_0}{dx} \\
a_1(\vartheta) \frac{df_1}{dx}
\end{bmatrix},
\]

(9)

where

\[
a_i(\vartheta) = s_i \cos^2 \vartheta + \omega_i \varrho_i \sin^2 \vartheta, \quad i = 0, 1.
\]

(10)

One can show [9] that the probability density \( p(\xi, \vartheta) = [p_0(\vartheta), p_1(\vartheta)] \) satisfies the adjoint equation \( \mathcal{L}^* p = 0 \) or the linear system

\[
(a_0(\vartheta)p_0)' = -\lambda_0 p_0 + \lambda_1 p_1,
(a_1(\vartheta)p_1)' = \lambda_0 p_0 - \lambda_1 p_1
\]

(11)

with periodic and normalization conditions

\[
p_i(0) = p_i(\pi);
\]

\[
\int_0^\pi p_i(\vartheta) d\vartheta = \frac{\lambda_1 - i}{\lambda_0 + \lambda_1}, \quad i = 0, 1.
\]

(12)

The fact that the system (11) has first integral

\[
a_0(\vartheta)p_0(\vartheta) + a_1(\vartheta)p_1(\vartheta) = c
\]

(13)

allows us to find an explicit solution of (11)

\[
p_i(\vartheta) = \frac{c\lambda_1 - i}{a_i(\vartheta)} \left\{ K_i \exp \left[ -\int_0^\vartheta \eta(\vartheta') d\vartheta' \right] + \int_0^\vartheta \frac{d\vartheta'}{a_{1-i}(\vartheta')} \exp \left[ -\int_0^{\vartheta'} \eta(\vartheta'') d\vartheta'' \right] \right\},
\]

(14)

where

\[
\eta(\vartheta) = \frac{\lambda_0}{a_0(\vartheta)} + \frac{\lambda_1}{a_1(\vartheta)},
\]

(15)

\[
K_i = \int_0^\pi \frac{d\vartheta'}{a_{1-i}(\vartheta')} \exp \left[ -\int_0^\vartheta \eta(\vartheta'') d\vartheta'' \right] \frac{1 - \exp \left[ -\int_0^\vartheta \eta(\vartheta') d\vartheta' \right]}{1 - \exp \left[ -\int_0^\vartheta \eta(\vartheta') d\vartheta' \right]}, \quad i = 0, 1,
\]

(16)

\[
c^{-1} = \sum_{i=0}^1 \lambda_1 - i \int_0^\pi \left[ K_i \exp \left[ -\int_0^\vartheta \eta(\vartheta') d\vartheta' \right] + \int_0^\vartheta \frac{d\vartheta'}{a_{1-i}(\vartheta')} \exp \left[ -\int_0^{\vartheta'} \eta(\vartheta'') d\vartheta'' \right] \right] \frac{d\vartheta}{a_i(\vartheta)}.
\]

(17)
3. Calculation of the Lyapunov exponent

The Lyapunov exponent $\gamma$ (the inverse of the localization length $\ell$) is a non-random quantity which determines the asymptotic growth or decay of the solution. If we denote by $L$ the total length of the rod, then by definition

$$\gamma \equiv \lim_{L \to \infty} \frac{\ln r(L)}{L} = \lim_{L \to \infty} \frac{1}{L} \int_0^L \frac{1}{2} (s(\xi_u) - \omega^2 \varrho(\xi_u)) \sin 2\vartheta \, du$$

$$= \frac{1}{2} \langle (s(\xi) - \omega^2 \varrho(\xi)) \sin 2\vartheta \rangle = \frac{1}{2} (s_0 - \omega^2 \varrho_0) \int_0^\pi p_0(\vartheta) \sin 2\vartheta \, d\vartheta$$

$$+ \frac{1}{2} (s_1 - \omega^2 \varrho_1) \int_0^\pi p_1(\vartheta) \sin 2\vartheta \, d\vartheta,$$

where the limit holds with probability one. Using (14), one can calculate the Lyapunov exponent exactly. However, for the sake of analytical tractability we will find asymptotic values of $\gamma$ for long and short waves. To this end, we analyse the dependence of $p_0(\vartheta)$ and $p_1(\vartheta)$ involved in (18) on $\omega$. Figure 2 shows the dependence of the probability density function $p_0(\vartheta)$ on $\omega$ for small and large values of the frequency. It has a maximum at the points where the phase $\vartheta$ (5) has slowest rate of change. Therefore, when $\omega \ll 1$ then $p_0(\vartheta)$ is concentrated around $\pi/2$ where $\cos \vartheta$ in (5) is small, while if $\omega \gg 1$ then $p_0(\vartheta)$ is located near zero or $\pi$ to minimize $\sin \vartheta$ in (5). We will use these results below to find the asymptotics of $\gamma$.

3.1 Long wave asymptotics of the Lyapunov exponent

The dependence of the probability density functions in figure 2 suggests that its asymptotic behaviour for small values of $\omega$ can be found from the Laplace method. Using this and the formulas in the appendix, we obtain

$$K_i \sim \frac{s_i}{\lambda_i s_0 + \lambda_1 s_1}, \quad i = 0, 1, \quad \omega \to 0.$$  

Figure 2. Dependence of the probability density function $p_0(\vartheta)$ (14) on the excitation frequency $\omega$ for $\varrho_0 = \varrho_1 = 1$ and $3s_0 = s_1 = 3, 2\lambda_0 = \lambda_1 = 2$. 

\[\int_0^\pi \frac{d\vartheta}{a_i(\vartheta)} \exp \left[-\int_0^\theta \eta(\vartheta') d\vartheta'\right] \sim \frac{s_{1-i}}{\lambda_0 s_1 + \lambda_1 s_0}, \quad i = 0, 1, \quad \omega \to 0. \quad (20)\]

\[\int_0^\pi \frac{d\vartheta}{a_i(\vartheta)} \int_0^\theta \frac{d\vartheta'}{a_{1-i}(\vartheta'')} \exp \left[-\int_\vartheta^{\theta'} \eta(\vartheta'') d\vartheta''\right] \sim \frac{\pi}{\omega(\lambda_0 + \lambda_1)\sqrt{\langle s \rangle \langle \varrho \rangle}}, \quad i = 0, 1, \quad (21)\]

where

\[\langle s \rangle = \frac{s_0 \lambda_1 + s_1 \lambda_0}{\lambda_0 + \lambda_1}, \quad \langle \varrho \rangle = \frac{\varrho_0 \lambda_1 + \varrho_1 \lambda_0}{\lambda_0 + \lambda_1}\]

are the average values of the elastic compliance and density, respectively. Finally, from (18) we obtain the asymptotic value of the Lyapunov exponent

\[\gamma(\omega) \sim \omega^2 \frac{\lambda_0 \lambda_1}{4} \frac{(s_0 \varrho_1 - s_1 \varrho_0)^2}{\langle \varrho \rangle \langle s \rangle}, \quad \omega \to 0. \quad (23)\]

### 3.2 Short wave asymptotics of the Lyapunov exponent

Asymptotics of integrals in (14) for large values of \( \omega \) can easily be found using integration by parts and the integrals in the appendix. Thus,

\[K_i \sim \frac{\sqrt{s_i \varrho_i}}{\lambda_0 \sqrt{s_1 \varrho_1} + \lambda_1 \sqrt{s_0 \varrho_0}}, \quad i = 0, 1, \quad \omega \to \infty \quad (24)\]

and

\[c \sim \frac{\omega}{\pi} \frac{\lambda_0 \sqrt{s_1 \varrho_1} + \lambda_1 \sqrt{s_0 \varrho_0}}{\lambda_0 + \lambda_1}, \quad \omega \to \infty. \quad (25)\]

Evaluation of the integrals in (18) gives

\[\gamma(\omega) \sim \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \left( \frac{s_1 \varrho_0}{s_0 \varrho_1} \right)^{1/8} - \left( \frac{s_0 \varrho_1}{s_1 \varrho_0} \right)^{1/8} \right]^2, \quad \omega \to \infty. \quad (26)\]

Figure 3 shows typical dependence of the Lyapunov exponent \( \gamma \) on the frequency \( \omega \). It does not exhibit oscillating behaviour due to the infinite number of alternating rods (cf. [3]). Unlike the random Schrödinger equation, waves in a random rod become more localized for high frequencies.

![Figure 3](image-url)
4. Reflectionless propagation

It is useful to rewrite formulas (23) and (26) using the impedance notation. By definition, the impedance $Z_i, i = 0, 1,$ of each rod is equal to

$$Z_i = \frac{\rho_i}{s_i}, \quad i = 0, 1.$$  \hspace{1cm} (27)

With this notation, asymptotic formulas for the Lyapunov exponent take the form

$$\gamma(\omega) \sim \frac{\omega^2}{4} \frac{\lambda_0 \lambda_1}{(\lambda_0 + \lambda_1)^2} \frac{\langle \omega \rangle \langle s \rangle}{(Z_0 - Z_1)^2}, \quad \omega \to 0$$  \hspace{1cm} (28)

and

$$\gamma(\omega) \sim \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} \left[ \left( \frac{Z_0}{Z_1} \right)^{1/8} - \left( \frac{Z_1}{Z_0} \right)^{1/8} \right]^2, \quad \omega \to \infty.$$  \hspace{1cm} (29)

Then when the impedances of the two rods are equal, $Z_0 = Z_1,$ the Lyapunov exponent equals zero. In this case waves in the composite rod propagate without reflection. Indeed, consider an infinite elastic rod consisting of only two semi-infinite homogeneous elastic rods with parameters $\rho_0, s_0$ and $\rho_1, s_1$ (figure 4). Propagation of waves in both rods is described by the wave equation

$$\frac{\partial^2 u_i}{\partial t^2} = v_i^2 \frac{\partial^2 u_i}{\partial x^2}, \quad v_i = \frac{1}{\sqrt{\rho_i s_i}}, \quad i = 0, 1,$$  \hspace{1cm} (30)

for $x > 0$ and $x < 0$ with the following boundary and initial conditions at $x = 0$

$$u_0(0, t) = u_1(0, t), \quad \frac{1}{s_0} \frac{\partial u_0(0, t)}{\partial x} = \frac{1}{s_1} \frac{\partial u_1(0, t)}{\partial x};$$

$$u_0(x, 0) = f\left(-\frac{x}{v_0}\right), \quad \frac{\partial u_0(x, 0)}{\partial t} = f_1\left(-\frac{x}{v_0}\right),$$  \hspace{1cm} (31)

where $u_0(x, t) = f(t - x/v_0)$ is the displacement in the left rod for $x < 0$ and $t \leq 0.$

The solution of equations (30) for $t > 0$ with conditions (31) have the form

$$u_0(x, t) = f(t - x/v_0) + \frac{Z_0 - Z_1}{Z_0 + Z_1} f(t + x/v_0), \quad x < 0,$$

$$u_1(x, t) = \frac{2Z_0}{Z_0 + Z_1} f(t - x/v_1), \quad x > 0.$$  \hspace{1cm} (32)

Thus, waves in a coupled rod propagate without reflection in the case of matching impedances $Z_0 = Z_1.$ This result remains true for multi-coupled rods as well.

Finally, we note that for the long waves the rod can be considered as homogeneous with effective parameters (22) if the propagation distance is significantly shorter than the localization length $\ell.$ Equation (2) in this case can be approximated by the wave equation with effective

![Figure 4](https://example.com/figure4.png)

Figure 4. An elastic rod composed of two semi-infinite perfectly connected rods with parameters $\rho_0, s_0$ and $\rho_1, s_1,$ where $\rho$ is the density and $s$ is the elastic compliance, respectively.
coefficients [10]

\[ \frac{1}{\langle \rho \rangle \langle s \rangle} \frac{d^2 X}{dx^2} + \omega^2 X = 0. \]  

(33)

From here we derive the effective velocity

\[ \overline{v} = \frac{1}{\sqrt{\langle \rho \rangle \langle s \rangle}} \]  

(34)

which is related to the average wave velocity

\[ \langle v \rangle = \frac{v \lambda_1 + v \lambda_0}{\lambda_0 + \lambda_1} \]  

(35)

as follows

\[ \langle v \rangle = \overline{v} \sqrt{1 + \frac{\lambda_0 \lambda_1}{(\lambda_0 + \lambda_1)^2} \frac{s_0 s_1 (\sqrt{Z_0} - \sqrt{Z_1})^2}{v^2}}. \]  

(36)

Therefore, the effective velocity is always less than the average one and can be equal in the case of matched impedances.

5. Conclusions

We have considered propagation of longitudinal waves in an elastic composite rod consisting of rods with alternating properties and random length. The use of the phase-amplitude representation of the solution allows us to reduce the problem to a system of two linear equation with respect to the joint probability distribution function whose solution is found exactly. Then we calculate exactly the Lyapunov exponent (the inverse of the localization length) and explicitly find its short and long wave asymptotics. Analysis of the asymptotic formulas shows that longitudinal waves in a composite rod propagate without reflection in the case of matched impedances of the adjacent rods. The methods used in the paper also allow us to analyse the variance of fluctuations of the Lyapunov exponent.

Acknowledgment

The author has the pleasant duty to thank Prof. S.A. Molchanov for encouragement and valuable discussions.

Appendix

Below we give the values of some integrals used in the paper.

\[ \int_0^\pi \eta(\theta) d\theta \]  

where

\begin{align*}
\eta(\theta) &= \frac{\lambda_0}{s_0 \cos^2 \vartheta + \omega^2 \varrho_0 \sin^2 \vartheta} + \frac{\lambda_1}{s_1 \cos^2 \vartheta + \omega^2 \varrho_1 \sin^2 \vartheta}, \\
&= \frac{\lambda_0}{\omega \sqrt{\varrho_0 s_0}} \left[ \frac{\pi}{2} - \arctan \frac{\cot \vartheta}{\omega \sqrt{\varrho_0}} \right] + \frac{\lambda_1}{\omega \sqrt{\varrho_1 s_1}} \left[ \frac{\pi}{2} - \arctan \frac{\cot \vartheta}{\omega \sqrt{\varrho_1}} \right].
\end{align*}
\[
\int_\vartheta^\vartheta' \eta(\vartheta'') d\vartheta'' = \frac{\lambda_0}{\omega \sqrt{\varrho_0 s_0}} \left[ \frac{\pi}{2} + \arctan \frac{\cot \vartheta'}{\omega \sqrt{\varrho_0}} \sqrt{s_0} \right] \arctan \frac{\cot \vartheta'}{\omega \sqrt{s_0}} - \arctan \frac{\cot \vartheta}{\omega \sqrt{s_0}} \right] + \frac{\lambda_1}{\omega \sqrt{\varrho_1 s_1}} \left[ \frac{\pi}{2} + \arctan \frac{\cot \vartheta'}{\omega \sqrt{\varrho_1}} \sqrt{s_1} \right] \arctan \frac{\cot \vartheta'}{\omega \sqrt{s_1}} - \arctan \frac{\cot \vartheta}{\omega \sqrt{s_1}} \right].
\]

References