A Multivariate Normal Law for Turing's Formulae *

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Abstract

This paper establishes a sufficient condition for Turing's formulae of various orders to have asymptotic multivariate normality. As an application, a consistent estimator of the tail under a discrete power tail model is also described.

1 Introduction.

Consider the population of all birds in the world with all of its different species. Suppose a random sample of $n = 2000$ is taken from the population and $M = 30$ different bird species are found in the sample. Since the total number of bird species in the population must exceed 30, one may be interested in the total population proportion of birds belonging to the species other than the 30 observed in the sample. Could this proportion be reasonably estimated? Good (1953) introduced a nonparametric estimator, credited largely to Alan Turing, for just that proportion in question. The formula is known as Turing's formula. Turing's formula suggests that, if among the $M = 30$ different species in the sample N_1 species include exactly one bird each, say $N_1 = 10$, then $N_1/n = 10/2000 = 0.005$ is a reasonable estimate for the proportion in question. Turing's formula has an intriguing implication: the total proportion associated with species not observed in the sample can be nonparametrically estimated (albeit subject to some error). This

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almost-anti-intuitive implication has inspired much research both in applications and in theory. One area which has seen much research related to Turing's formula includes the problems of estimating population diversity indices. There are several different types of diversity indices that are commonly discussed in the literature, the total number of population species, Simpson's index and Shannon's entropy. There is a large volume of publications on estimation of total number of population species incorporating Turing's formula in some way. Notable publications there include Chao (1984), Chao and Lee (1992), Chao and Bunge (2002), Mao and Lindsay (2005), Wang and Lindsay (2005), and Zhang and Stern (2009) among many others. Zhang and Zhou (2010) discussed the estimation problem of Simpson's diversity and its generalized forms. On the estimation of Shannon's entropy, notable research includes Chao and Shen (2003), Vu, Yu and Kass (2007), and Zhang (2012). However deviating from the themes of the above mentioned references, the focus of the current paper is on the large sample distributional characteristics of Turing's formula and its higher order relatives.

Consider a multinomial distribution with its countably infinite number of prescribed categories indexed by $K = \{k; k = 1, \cdots\}$ and its category probabilities denoted by $\{p_k\}$, satisfying $0 < p_k < 1$ for all k and $\sum p_k = 1$, where the sum without index is over all k as is observed in the subsequent text unless otherwise stated. Let the category counts in an *iid* sample of size n from the underlying population be denoted by $\{X_k; k \geq 1\}$ and its observed values by $\{x_k; k \geq 1\}$. For a given sample, there are at most n non-zero x_k's. Let, for every integer r, $1 \leq r \leq n$,

$$N_r = \sum 1_{[X_k=r]}, \quad T_r = \binom{n}{r-1}\binom{n}{r}^{-1} N_r = \frac{r}{n-r+1} N_r, \quad \textit{and} \quad \pi_{r-1} = \sum p_k 1_{[X_k=r-1]}.$$

N_r and π_{r-1} may be thought of as, respectively, the number of categories in the population that are represented exactly r times in the sample and the total probability associated with all the categories that are represented exactly $r-1$ times in the sample. T_r may be thought of as an estimator of π_{r-1}. T_r is also known as Turing's formula of the r^{th} order introduced by Good (1953). Perhaps the most interesting case among all Turing's formulae of different orders is T_1, known as

just Turing's formula, as an estimator of π_0. Given a sample, π_0 represents the total probability associated with categories not observed in the sample, which is also the probability that the next observation will belong to a category previously unseen. Since the multinomial model is essentially nonparametric, the fact that something could be said about the total probability associated with unobserved categories is somewhat anti-intuitive. The statistical properties of Turing's formula were largely unknown until Robbins (1968) gave an interpretation in terms of bias. Another fifteen years would pass before Esty (1983) gave a sufficient condition for the asymptotic normality of $T_1 - \pi_0$. In recent years, research on Turing's formula has been revitalized. Zhang and Huang (2007) gave another interpretation of Turing's formula and proposed an improved version of the formula which essentially eliminated all the bias of Turing's original formula. Zhang and Huang (2008) gave a sufficient condition for the normality of Turing's formula which supports a non-empty class of fixed distributions. Zhang and Zhang (2009) gave a necessary and sufficient condition for the normality of Turing's formula. Lijoi, Mena and Prünster (2007) offered an discussion in Bayesian perspective. However all the works thus far are on Turing's formula of the first order. Prior to this paper, the distributional properties of high order Turing's formulae are unknown.

The objective of this paper is to establish the asymptotic multivariate normality of several Turing's formulas of various orders under certain conditions. The main line of the proofs in this paper largely follows that of Esty (1983) or Zhang and Huang (2008) which involves direct evaluations of the characteristic functions of properly normalized Turing's formulas of various orders. Several key preliminary results are given in Section 2, followed by the univariate normality results for Turing's formulae in Section 3, and followed by the multivariate normality results in Section 4. In Section 5, a brief discussion of how the established results could be used to derive a consistent estimator for the tail of a discrete probability distribution with a power decaying tail. Many of the lengthy proofs for the results throughout the paper are given in Section 6.

2 Preliminary Results.

Let $K_1 = \{1\}$ and $K_2 = \{2, \cdots\}$. For any $k \in K = K_1 \cup K_2$, let

$$f_k(x) = \begin{cases} p_k & x = r-1, \\ -r/(n-r+1) & x = r, \\ 0 & 0 \le x \le r-2 \text{ or } x \ge r+1, \end{cases} \tag{2.1}$$

and $Z = \sum f_k(X_k)$. The objective is to derive the asymptotic behavior of $Zg(n)$, where $g(n)$ is a function of n satisfying

$$g(n) = O(n^{1-2\delta}) \tag{2.2}$$

for some $\delta \in (0, 1/4)$, in terms of the limit of its characteristic function, $E[\exp(isZg(n))]$. Let $Z = Z_1 + Z_2$, where $Z_1 = \sum_{K_1} f_k(X_k)$ and $Z_2 = \sum_{K_2} f_k(X_k)$. Lemma 2.1 below is a well-known fact and Lemma 2.2 is due to Bartlett (1938).

Lemma 2.1 *Let $\{X_k\}$ be the counts of observations in category k, $k = 1, 2, \cdots$, in an iid sample under the multinomial model with probability distribution $\{p_k\}$. Then*

$$P(X_k = x_k; k = 1, \cdots) = P(Y_k = x_k; k = 1, \cdots | \sum Y_k = n)$$

where $\{Y_k\}$ are independent Poisson random variables with mean np_k.

Lemma 2.2 *Let (U, V) be a two-dimensional random vector with U integer valued. Then*

$$E(\exp(ivV|U = n)) = (2\pi P(U = n))^{-1} \int_{-\pi}^{\pi} E[\exp(iu(U-n) + ivV)]du.$$

Thus $E(\exp(isZg(n)))$ is

$$\left(2\pi P\left(\sum Y_k = n\right)\right)^{-1} \int_{-\pi}^{\pi} E\left[\exp\left(iu\sum(Y_k - np_k) + isZg(n)\right)\right] du.$$

By Stirling's formula, $(2\pi n)^{1/2} P(\sum Y_k = n) \to 1$. Therefore it suffices to evaluate the limit of

$$H_n(s) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} E[\exp(iu\sum(Y_k - np_k) + isZg(n))]du,$$

or letting $t = un^{1/2}$,

$$H_n(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} 1_{[|t|<\pi\sqrt{n}]} E[\exp(i(n)^{-1/2} t \sum (Y_k - np_k) + isZg(n))] dt. \tag{2.3}$$

Let

$$\begin{aligned} h_n &= 1[|t| < \pi\sqrt{n}] E[\exp(i(n)^{-1/2} t \textstyle\sum (Y_k - np_k) + isZg(n)] \\ h_{n1} &= 1[|t| < \pi\sqrt{n}] E[\exp(i(n)^{-1/2} t (Y_1 - np_1) + isZ_1 g(n)] \qquad (2.4) \\ h_{n2} &= 1[|t| < \pi\sqrt{n}] E[\exp(i(n)^{-1/2} t \textstyle\sum_{K_2} (Y_k - np_k) + isZ_2 g(n)], \end{aligned}$$

$$H_n(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_n dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_{n1} h_{n2} dt. \tag{2.5}$$

The first task is to allow the limit operator to change place with the integral operator, *i.e.*, to show $\lim H_n(s) = \frac{1}{\sqrt{2\pi}} \int \lim h_n dt$ where $\lim = \lim_{n\to\infty}$ as is observed elsewhere in the subsequent text. The key element to support this exchange is

$$\lim \int |\overline{h}_{n1}| dt = \int \lim |\overline{h}_{n1}| dt, \tag{2.6}$$

where

$$|\overline{h}_{n1}| = 1_{[|t|\le\pi\sqrt{n}]} \Big\{ \exp(-itn^{1/2} p_1) \exp[np_1 (e^{itn^{-1/2}} - 1)]$$

$$+2[(np_1)^{r-1} \exp(-np_1)/(r-1)! + (np_1)^r \exp(-np_1)/r!] \}$$

is an upper bound for $|h_{n1}|$ and hence, since $|h_{n2}| \le 1$ implies $|h_n| \le |h_{n1}|$, an upper bound for $|h_n|$. The proof of (2.6) is given by Zhang and Huang (2008) for a special case of $r = 1$, however the proof is also valid for any $r \ge 1$.

By (2.6) and the extended dominated convergence theorem of Pratt (1960), the following lemma is established.

Lemma 2.3 *Let h_n and H_n be as defined in (2.3) and (2.5) respectively. Then*

$$\lim H_n = \frac{1}{\sqrt{2\pi}} \int \lim h_n dt.$$

For each k, it can be verified that, letting

$$\begin{aligned} B_k &= \exp(-itp_k n^{1/2})[\exp(np_k(\exp(itn^{-1/2})-1))] \\ C_k &= \exp(-itp_k n^{1/2})(\exp(isp_k g(n))-1)\exp(it(r-1)n^{-1/2})\exp(-np_k)\tfrac{(np_k)^{r-1}}{(r-1)!} \\ D_k &= \exp(-itp_k n^{1/2})[\exp(-isr(n-r+1)^{-1}g(n))-1]\exp(itrn^{-1/2})\exp(-np_k)\tfrac{(np_k)^{r}}{r!}, \end{aligned}$$

and $E_k = C_k + D_k$, $h_n = \prod(B_k + E_k)$ for all $t \in 0 \pm \pi\sqrt{n}$. The objective is to evaluate $\lim \prod(B_k + E_k)$.

The following two lemmas are given by Esty (1983) where "$\sim$" is equality in the limit as is observed elsewhere in the subsequent text.

Lemma 2.4 *Let $\{\beta_k\}$ and $\{\epsilon_k\}$ be two sequences of complex numbers, and M_n be a sequence of subsets of K, indexed by n. If*

1. $\prod_{M_n} \beta_k \sim \beta$,
2. $(\sum_{M_n} \epsilon_k) \sim \epsilon$,
3. $\beta_k \sim 1$ *uniformly,*
4. $\epsilon_k \sim 0$ *uniformly,*
5. *there exists a constants, δ_1 such that, $\sum_{M_n} |\beta_k - 1| \leq \delta_1$, and*
6. *there exists a constants, δ_2 such that, $\sum_{M_n} |\epsilon_k| \leq \delta_2$;*

then

$$\prod_{M_n} (\beta_k + \epsilon_k) \sim \beta e^{\epsilon}$$

where β and ϵ may also depend on n.

Lemma 2.5 *For all $k \in K$, $B_k = \exp((-t^2/2)p_k + O(t^3 p_k n^{-1/2}))$.*

The next lemma includes three trivial but useful facts.

Lemma 2.6 *1. For any complex number x satisfying $|x| < 1$, $|\ln(1+x)| \le \frac{|x|}{1-|x|}$.*

2. For any real number $x \in [0,1)$, $1 - x \ge \exp\left(-\frac{x}{1-x}\right)$.

3. For any real number $x \in (0, 1/2)$, $\frac{1}{1-x} < 1 + 2x$.

Proof. (1) By Taylor's formula, $|\ln(1+x)| = |\sum_{j=1}^{\infty}(-1)^{j+1}x^j/j| \le \sum_{j=1}^{\infty}|x|^j = |x|/(1-|x|)$. (2) The function $y = \frac{1}{1+t}e^t$ is strictly increasing over $[0,\infty)$, and has value 1 at $t = 0$. Therefore $\frac{1}{1+t}e^t \ge 1$ for $t \in [0,\infty)$. The desired inequality follows the change of variable $x = t/(1+t)$. (3) The proof is trivial. □

Consider a partition of the index set $K = I \cup II$ where

$$I = \{k; p_k \le r/n^{1-\delta^*}\} \quad \text{and} \quad II = \{k; p_k > r/n^{1-\delta^*}\}$$

where $\delta^* = \delta/(R+1)$ and δ is as in (2.2).

Lemma 2.7 *(a) $\sum_{II}|E_k| \to 0$; and (b) $\prod_{II}(B_k + E_k)/\prod_{II} B_k \to 1$.*

Proof. (a) $\sum_{II}|E_k| \le 2\sum_{II}[e^{-np_k}(np_k)^{(r-1)}/(r-1)! + e^{-np_k}(np_k)^r/r!]$. Since the derivative of $[e^{-np}(np)^{(r-1)}/(r-1)! + e^{-np}(np)^r/r!]$ with respect to p is negative for all $p \in (r/n, 1]$ (and therefore for all $p \in (r/n^{1-\delta^*}, 1]$), $[e^{-np_k}(np_k)^{(r-1)}/(r-1)! + e^{-np_k}(np_k)^r/r!]$ attains its maximum at $p_k = r/n^{1-\delta^*}$, for every $k \in II$, with value $e^{-rn^{\delta^*}}O(n^{r\delta^*})$. The total number of indices in II is less or equal to $n^{1-\delta^*}/r$. Therefore

$$\sum_{II}|E_k| \le 2[n^{1-\delta^*}/r][e^{-rn^{\delta^*}}O(n^{r\delta^*})] = (2/r)e^{-rn^{\delta^*}}O\left(n^{1+(r-1)\delta^*}\right) \to 0.$$

(b) By Lemma 2.5, $|B_k|$ is bounded away from zero, and by the fact that $\lim|E_k| = 0$ (and hence $\lim|E_k|/|B_k| = 0$), and by applying the first part of Lemma 2.6 with $x = E_k/B_k$, one has

$$|\ln[\textstyle\prod_{II}(B_k+E_k)/\prod_{II}B_k]| = \left|\sum_{II}\ln\left(1+\frac{E_k}{B_k}\right)\right| \le \sum_{II}\left|\ln\left(1+\frac{E_k}{B_k}\right)\right|$$

$$\le \sum_{II}\left(\frac{|E_k|}{|B_k|-|E_k|}\right) = O(\textstyle\sum_{II}|E_k|) \to 0.$$

□

The following is a sufficient condition under which many of the subsequent results are established.

Condition 2.1 *As* $n \to \infty$,

1. $\sum n^{r-2} g^2(n) p_k^r e^{-np_k} \to c_r \geq 0$,
2. $\sum n^{r-1} g^2(n) p_k^{r+1} e^{-np_k} \to c_{r+1} \geq 0$, *and*
3. $c_r + c_{r+1} > 0$.

Lemma 2.8 *Under Condition 2.1, all the conditions of Lemma 2.4 are satisfied with* $M_n = I$, $\beta_k = B_k$, $\beta = B = \lim \prod B_k$, $\epsilon_k = E_k$, *and* $\epsilon = E = \lim \sum E_k$.

The proof of Lemma 2.8 is given in Appendix. Lemma 2.4 and Lemma 2.8 give immediately the following corollary.

Corollary 2.1 *Under Condition 2.1,* $\prod_I (B_k + E_k) \sim \prod_I B_k \exp(\sum_I E_k)$.

Lemma 2.9 *Under Condition 2.1,* $\prod (B_k + E_k) \to Be^E$, *where* $B = \lim \prod B_k$ *and* $E = \lim \sum E_k$.

Proof.

$$\begin{aligned}
\prod (B_k + E_k) &= \prod_I (B_k + E_k) \prod_{II} (B_k + E_k) \sim \prod_I (B_k + E_k) \prod_{II} B_k \qquad \text{(by Lemma 2.7)} \\
&\sim \prod_I B_k (\exp \textstyle\sum_I E_k) \prod_{II} B_k \qquad \text{(by Lemma 2.8)} \\
&\sim \prod B_k (\exp \textstyle\sum E_k) \qquad \text{(by Lemma 2.7).}
\end{aligned}$$

□

3 Univariate Normality.

Theorem 3.1 *Let* $g(n)$ *be as in (2.2). Under Condition 2.1,*

$$g(n)(T_r - \pi_{r-1}) \xrightarrow{L} N\left(0, \frac{c_{r+1} + r c_r}{(r-1)!}\right).$$

Proof. Since $\lim \prod B_k = e^{-\frac{t^2}{2}}$, by (a) of Lemma 2.7 and (6.2),

$$\lim \sum E_k = -\frac{s^2}{2(r-1)!}\left[\lim \sum n^{r-1} g^2(n) p_k^{r+1} e^{-np_k} + r \lim \sum \frac{g^2(n) n^r p_k^r}{(n-r+1)^2} e^{-np_k}\right],$$

$$\lim H_n = \left(\frac{1}{\sqrt{2\pi}} \int e^{-\frac{t^2}{2}} dt\right) e^{-\frac{s^2}{2(r-1)!}\left[\lim \sum n^{r-1} g^2(n) p_k^{r+1} e^{-np_k} + r \lim \sum \frac{g^2(n) n^r p_k^r}{(n-r+1)^2} e^{-np_k}\right]}$$

$$= e^{-\frac{s^2}{2}\left[\frac{c_{r+1}}{(r-1)!} + \frac{r c_r}{(r-1)!}\right]}.$$

□

Consider the following condition:

Condition 3.1 *As* $n \to \infty$,

1. $\frac{g^2(n)}{n^2} E(N_r) \to \frac{c_r}{r!} \geq 0$,
2. $\frac{g^2(n)}{n^2} E(N_{r+1}) \to \frac{c_{r+1}}{(r+1)!} \geq 0$, *and*
3. $c_r + c_{r+1} > 0$.

Lemma 3.1 *Condition 2.1 and Condition 3.1 are equivalent.*

The proof of Lemma 3.1 is given in Appendix. Lemma 3.1 allows a re-statement of Theorem 3.1:

Theorem 3.2 *If there exists a* $g(n)$ *satisfying (2.2) and Condition 3.1, then*

$$\frac{n(T_r - \pi_{r-1})}{\sqrt{r^2 E(N_r) + (r+1) r E(N_{r+1})}} \xrightarrow{L} N(0,1).$$

Theorem 3.3 *If there exists a* $g(n)$ *satisfying (2.2) and Condition 3.1, then*

$$\frac{n(T_r - \pi_{r-1})}{\sqrt{r^2 N_r + (r+1) r N_{r+1}}} \xrightarrow{L} N(0,1).$$

The proof of Theorem 3.3 is given in Appendix.

It may be of interest to note that the results of Theorems 3.2 and 3.3 require no further knowledge of $g(n)$, *i.e.*, the knowledge of δ, other than its existence.

4 Multivariate Normality.

For every $k \in K$, any two constants a and b, and any two positive integers r_1 and r_2, let $f_k(x)$ in (2.1) be redefined as

$$f_k(x) = \begin{cases} ap_k & x = r_1 - 1, \\ -ar_1/(n - r_1 + 1) & x = r_1 \neq r_2 - 1, \\ bp_k & x = r_2 - 1 \neq r_1, \\ -br_2/(n - r_2 + 1) & x = r_2, \\ bp_k - ar_1/(n - r_1 + 1) & x = r_1 = r_2 - 1, \\ 0 & \text{elsewhere}, \end{cases} \tag{4.1}$$

and $Z = \sum f_k(X_k)$. The objective is to evaluate $\lim H_n(s) = (2\pi)^{-1/2} \int \lim h_n dt$ where $H_n(s)$ and h_n have the same forms as in (2.4) and (2.5) but with $f_k(x)$ redefined in (4.1). Two separate cases are to be considered: $r_1 < r_2 - 1$ and $r_1 = r_2 - 1$.

Let

$$\begin{aligned} B_k &= \exp(-itp_k n^{1/2})[\exp(np_k(\exp(itn^{-1/2}) - 1))] \\ C_k &= \exp(-itp_k n^{1/2})(\exp(isap_k g(n)) - 1)\exp(it(r_1 - 1)n^{-1/2})\tfrac{(np_k)^{r_1-1}}{(r_1-1)!}e^{-np_k} \\ D_k &= \exp(-itp_k n^{1/2})[\exp(-isar_1(n - r_1 + 1)^{-1}g(n)) - 1]\exp(itr_1 n^{-1/2})\tfrac{(np_k)^{r_1}}{r_1!}e^{-np_k} \\ F_k &= \exp(-itp_k n^{1/2})(\exp(isbp_k g(n)) - 1)\exp(it(r_2 - 1)n^{-1/2})\tfrac{(np_k)^{r_2-1}}{(r_2-1)!}e^{-np_k} \\ G_k &= \exp(-itp_k n^{1/2})[\exp(-isbr_2(n - r_2 + 1)^{-1}g(n)) - 1]\exp(itr_1 n^{-1/2})\tfrac{(np_k)^{r_2}}{r_2!}e^{-np_k} \\ A_k &= \exp(-itp_k n^{1/2})\{\exp[isg(n)(bp_k - a\tfrac{r_1}{n-r_1+1}) - 1]\}\exp(itr_1 n^{-1/2})\tfrac{(np_k)^{r_1}}{r_1!}e^{-np_k}. \end{aligned}$$

If $r_1 < r_2 - 1$, let $E_k = C_k + D_k + F_k + G_k$. If $r_1 = r_2 - 1$, let $E_k = C_k + A_k + G_k$. It can be verified that, in either case, $h_n = \prod(B_k + E_k)$ for all $t \in 0 \pm \pi\sqrt{n}$. The objective is to evaluate $\lim \prod(B_k + E_k)$.

Condition 4.1 *As* $n \to \infty$,

1. $\frac{g^2(n)}{n^2}E(N_{r_1}) \to \frac{c_{r_1}}{r_1!} \geq 0$,

2. $\frac{g^2(n)}{n^2}E(N_{r_1+1}) \to \frac{c_{r_1+1}}{(r_1+1)!} \geq 0$,

3. $c_{r_1} + c_{r_1+1} > 0$,

4. $\frac{g^2(n)}{n^2} E(N_{r_2}) \to \frac{c_{r_2}}{r_2!} \geq 0$,

5. $\frac{g^2(n)}{n^2} E(N_{r_2+1}) \to \frac{c_{r_2+1}}{(r_2+1)!} \geq 0$, *and*

6. $c_{r_2} + c_{r_2+1} > 0$.

Lemma 4.1 *For any two constants, a and b satisfying $a^2 + b^2 > 0$, assuming that $r_1 < r_2 - 1$ and that Condition 4.1 holds, then*

$$g(n)[a(T_{r_1} - \pi_{r_1-1}) + b(T_{r_2} - \pi_{r_2-1})] \xrightarrow{L} N(0, \sigma^2)$$

where $\sigma^2 = a^2 \frac{c_{r_1+1} + r_1 c_{r_1}}{(r_1-1)!} + b^2 \frac{c_{r_2+1} + r_2 c_{r_2}}{(r_2-1)!}$.

The proof of Lemma 4.1 is straight forward in light of the argument that led to Theorem 3.1.

Lemma 4.2 *For any two constants, a and b satisfying $a^2 + b^2 > 0$, assuming that $r_1 = r_2 - 1$ and that Condition 4.1 holds, then*

$$g(n)[a(T_{r_1} - \pi_{r_1-1}) + b(T_{r_2} - \pi_{r_2-1})] \xrightarrow{L} N(0, \sigma^2)$$

where $\sigma^2 = a^2 \frac{c_{r_1+1} + r_1 c_{r_1}}{(r_1-1)!} - 2ab \frac{c_{r_2}}{(r_1-1)!} + b^2 \frac{c_{r_2+1} + r_2 c_{r_2}}{(r_2-1)!}$.

The proof of Lemma 4.2 is also straight forward in light of the argument that led to Theorem 3.1, but with an additional non-vanishing term in the limit.

Let $\sigma_r^2 = r^2 E(N_r) + (r+1) r E(N_{r+1})$, $\rho_r(n) = -r(r+1)E(N_{r+1})/(\sigma_r \sigma_{r+1})$, $\rho_r = \lim \rho_r(n)$, $\hat{\sigma}_r^2 = r^2 N_r + (r+1) r N_{r+1}$, and $\hat{\rho}_r = \hat{\rho}_r(n) = -r(r+1)N_{r+1}/\sqrt{\hat{\sigma}_r^2 \hat{\sigma}_{r+1}^2}$.

Corollary 4.1 *Assume that $r_1 < r_2 - 1$ and that Condition 4.1 holds, then*

$$n\left[(T_{r_1} - \pi_{r_1-1})/\sigma_{r_1}, (T_{r_2} - \pi_{r_2-1})/\sigma_{r_2}\right]' \xrightarrow{L} MVN\left(0, I_{2\times 2}\right).$$

Corollary 4.2 *Assume that $r_1 = r_2 - 1$ and that Condition 4.1 holds, then*

$$n\left[(T_{r_1} - \pi_{r_1-1})/\sigma_{r_1}, (T_{r_2} - \pi_{r_2-1})/\sigma_{r_2}\right]' \xrightarrow{L} MVN\left(0, \begin{pmatrix} 1 & \rho_{r_1} \\ \rho_{r_1} & 1 \end{pmatrix}\right).$$

Remark 4.1 *Corollaries 4.1 and 4.2 suggest that, in $\{n[(T_r - \pi_{r-1})/\sigma_r]; r = 1, \cdots, R\}$, any two entries are asymptotically independent unless they are immediate neighbors in the series.*

Corollaries 4.1 and 4.2 immediately give the following theorem.

Theorem 4.1 *For any positive integer R, if Condition 3.1 holds for every r, $1 \le r \le R$, then*

$$n\left[(T_1 - \pi_0)/\sigma_1, \cdots, (T_R - \pi_{R-1})/\sigma_R\right]' \xrightarrow{L} MVN(0, \Sigma)$$

where $\Sigma = (a_{i,j})$ is a $R \times R$ covariance matrix with all the diagonal elements being $a_{r,r} = 1$ for $r = 1, \cdots, R$, the super-diagonal and the sub-diagonal elements being $a_{r,r+1} = a_{r+1,r} = \rho_r$ for $r = 1, \cdots, R-1$, and all the other off-diagonal elements being zeros.

Let $\hat{\Sigma}$ be the resulting matrix of Σ with ρ_r replaced by $\hat{\rho}_r(n)$ for all r. Let $\hat{\Sigma}^{-1}$ denote the inverse of $\hat{\Sigma}$ and $\hat{\Sigma}^{-1/2}$ denote any $R \times R$ matrix satisfying $\hat{\Sigma}^{-1} = \hat{\Sigma}^{-1/2}\hat{\Sigma}^{-1/2}$. With the consistency of these estimators already established in the proof of Theorem 3.3, Theorem 4.2 below follows Theorem 4.1.

Theorem 4.2 *For any positive integer R, if Condition 3.1 holds for every r, $1 \le r \le R$, then*

$$n\hat{\Sigma}^{-1/2}\left[(T_1 - \pi_0)/\hat{\sigma}_1, \cdots, (T_R - \pi_{R-1})/\hat{\sigma}_R\right]' \xrightarrow{L} MVN(0, I_{R\times R}).$$

5 An Application.

An interesting special case of discrete distribution is that of $\{p_k\}$ following a discrete power law, as known as a Pareto law, in the tail, *i.e.*,

$$p_k = Ck^{-\lambda} \tag{5.1}$$

for all $k \geq k_0$ where $C > 0$ and $\lambda > 1$ are unknown parameters describing the tail of the probability distribution at and beyond an unknown positive integer k_0, while for each $k < k_0$, p_k conforms to no particular parametric forms. This partially parametric probability model is also known as one of the "power tail models". Suppose it is of interest to estimate C and λ. The results of last section could be used to develop a consistent estimation procedure as proposed below.

Lemma 5.1 *Under the model in (5.1), Condition 4.1 holds.*

Proof. Letting $\delta = (4\lambda)^{-1}$ in (2.2), it can be verified that $n^{r-2}g^2(n)\sum p_k^r e^{-np_k} \to c_r > 0$ for every integer $r > 0$. □

Corollary 5.1 *Under the model in (5.1), the results of both Theorems 4.1 and 4.2 hold.*

Let $\delta = (4\lambda)^{-1}$ in (2.2). Under the power tail model (5.1), for every $r > 0$, let every p_k in π_{r-1} be replaced by $p_k = Ck^{-\lambda}$ and denote the resulting expression as π^*_{r-1}, *i.e.*, $\pi^*_{r-1} = C\sum k^{-\lambda}1[y_k = r-1]$. Then $\pi^*_r - \pi_r = \sum_{k=1}^{k_0-1} 1[y_k = r-1](p_k - Ck^{-\lambda})$, and the sum is of finite terms. Since, for every k, both $E(1[y_k = r-1])$ and $Var(1[y_k = r-1])$ converge to zero exponentially, and hence $E(g_n(\delta)1[y_k = r-1])$ and $Var(g_n(\delta)1[y_k = r-1])$ converge to zero, which implies that $g_n(\delta)1[y_k = r-1] \xrightarrow{p} 0$, which in turn implies that $\pi^*_r - \pi_r \xrightarrow{p} 0$ for every r. This argument leads to the following two lemmas:

Lemma 5.2 *Under the power tail model in (5.1),*

$$n\left[(T_1 - \pi^*_0)/\sigma_1, \cdots, (T_R - \pi^*_{R-1})/\sigma_R\right]' \xrightarrow{L} MVN(0, \Sigma)$$

where Σ is as in Lemma 4.1.

Lemma 5.3 *Under the power tail model in (5.1),*

$$n\hat{\Sigma}^{-1/2}\left[(T_1 - \pi^*_0)/\hat{\sigma}_1, \cdots, (T_R - \pi^*_{R-1})/\hat{\sigma}_R\right]' \xrightarrow{L} MVN(0, I_{R\times R}). \tag{5.2}$$

where $\hat{\Sigma}^{-1/2}$ is as in Lemma 4.2.

The asymptotic likelihood given by (5.2) depends on the two model parameters, λ and C, only via $\pi^*_{r-1} = C\sum 1[y_k = r-1]k^{-\lambda}$, $r = 1, \cdots, R$.

Definition 5.1 *If there exists a pair of values, $C = \hat{C}$ and $\lambda = \hat{\lambda}$, which maximizes the likelihood given by (5.2), then $(\hat{C}, \hat{\lambda})$ is said to be an asymptotic maximum likelihood estimator (AMLE) of (C, λ).*

Given a sample, an $AMLE$ minimizes

$$\ell\ell_n = \ell\ell_n(C,\lambda) = \left(T_1 - \pi^*_0, \cdots, T_R - \pi^*_{R-1}\right)\hat{\Sigma}^{-1}_R\left(T_1 - \pi^*_0, \cdots, T_R - \pi^*_{R-1}\right)'$$

where

$$\hat{\Sigma}^{-1}_R = n\hat{\Sigma}^{-1}\begin{pmatrix} 1/\hat{\sigma}^2_1 & & \\ & \ddots & \\ & & 1/\hat{\sigma}^2_R \end{pmatrix}$$

and the last displayed $R \times R$ matrix above has non-zero elements only on the diagonal.

Finally the following theorem can be verified.

Theorem 5.1 *Under the power tail model in (5.1), the AMLE of (C, λ) is unique for a sufficiently large n and is consistent.*

6 Appendix.

6.1 Proof of Lemma 2.8.

All six conditions in Lemma 2.4 need to be checked.

(3) is true because

$$B_k = \exp(-(t^2/2)p_k)\exp(O((t^3/\sqrt{n})p_k))),$$

and p_k and $p_k/\sqrt{n}$ are uniformly bounded by $\frac{r}{n^{1-\delta^*}}$ and $\frac{r}{\sqrt{n}n^{1-\delta^*}}$ respectively.

For (1), since $\sum_I p_k \to 0$,

$$\prod_I B_k = \exp(-(t^2/2)\sum_I p_k)\exp(O((t^3/\sqrt{n})\sum_I p_k))) \to 1.$$

For (4), it suffices to show that $|C_k|$ and $|D_k|$ respectively converge to zero uniformly. First for all $k \in I$, $\exp(-itp_k\sqrt{n}) \to 1$ uniformly since $p_k\sqrt{n} \leq \frac{\sqrt{n}}{g(n)n^{\delta^*}} = O(n^{-1/2+\delta^*}) \to 0$ uniformly. Second, $\exp(it(r-1)n^{-1/2}) \to 0$ and $\exp(itrn^{-1/2}) \to 0$ uniformly. Third, $\exp(-np_k) \leq 1$ uniformly. By Taylor's expansion and for sufficiently large n,

$$\begin{aligned}
[\exp(isp_kg(n)) - 1]\,\tfrac{(np_k)^{r-1}}{(r-1)!} &= \left(isg(n)p_k - \tfrac{s^2g^2(n)p_k^2}{2!} - O\left(s^3g^3(n)p_k^3\right)\right)\tfrac{(np_k)^{r-1}}{(r-1)!} \\
&= \tfrac{isn^{r-1}g(n)p_k^r}{(r-1)!} - \tfrac{s^2n^{r-1}g^2(n)p_k^{r+1}}{2!(r-1)!} - O\left(s^3n^{r-1}g^3(n)p_k^{r+2}\right) \\
&\leq \left|\tfrac{isn^{r-1}g(n)p_k^r}{(r-1)!}\right| + \left|\tfrac{s^2n^{r-1}g^2(n)p_k^{r+1}}{2!(r-1)!}\right| + \left|O\left(s^3n^{r-1}g^3(n)p_k^{r+2}\right)\right| \\
&\leq \tfrac{sr^r}{(r-1)!}n^{-2\delta+\frac{r}{R+1}\delta} + \tfrac{s^2r^{r+1}}{2!(r-1)!}n^{-4\delta+\frac{r+1}{R+1}\delta} + O\left(n^{-6\delta+\frac{r+2}{R+1}\delta}\right) \to 0
\end{aligned}$$

uniformly.

Similarly, it is easily checked that

$$\begin{aligned}
\left[\exp(-isr(n-r+1)^{-1}g(n)) - 1\right]\tfrac{(np_k)^r}{r!} &= \left(-\tfrac{isrg(n)}{n-r+1} - \tfrac{s^2r^2g^2(n)}{2!(n-r+1)^2} + O\left(\tfrac{s^3r^3g^3(n)}{3!(n-r+1)^3}\right)\right)\tfrac{(np_k)^r}{r!} \\
&= -\tfrac{isrg(n)n^rp_k^r}{r!(n-r+1)} - \tfrac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2} + O\left(\tfrac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right) \\
&\leq \left|\tfrac{isrg(n)n^rp_k^r}{r!(n-r+1)}\right| + \left|\tfrac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2}\right| + \left|O\left(\tfrac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right)\right| \\
&\leq \tfrac{sr^r}{(r-1)!}\tfrac{n}{n-r+1}n^{-2\delta+\frac{r}{R+1}\delta} + \tfrac{s^2r^{r+2}}{2!r!}\tfrac{n^2}{(n-r+1)^2}n^{-4\delta+\frac{r}{R+1}\delta} + O\left(\tfrac{n^3}{(n-r+1)^3}n^{-6\delta+\frac{r}{R+1}\delta}\right) \to 0
\end{aligned}$$

uniformly. Therefore $E_k \to 0$ uniformly.

For (2) and (6),

$$
\begin{aligned}
E_k &= e^{-np_k}\exp(-itp_k\sqrt{n})\exp(it(r-1)n^{-1/2})\left[\frac{isn^{r-1}g(n)p_k^r}{(r-1)!}-\frac{s^2n^{r-1}g^2(n)p_k^{r+1}}{2!(r-1)!}\right.\\
&\quad\left.-O\left(s^3n^{r-1}g^3(n)p_k^{r+2}\right)\right]\\
&\quad+e^{-np_k}\exp(-itp_k\sqrt{n})\exp(itrn^{-1/2})\left[-\frac{isrg(n)n^rp_k^r}{r!(n-r+1)}-\frac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2}+O\left(\frac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right)\right]\\
&= e^{-np_k}\exp(-itp_k\sqrt{n})\exp(it(r-1)n^{-1/2})\left[\frac{isn^{r-1}g(n)p_k^r}{(r-1)!}-\frac{s^2n^{r-1}g^2(n)p_k^{r+1}}{2!(r-1)!}\right.\\
&\quad\left.-O\left(s^3n^{r-1}g^3(n)p_k^{r+2}\right)\right]\\
&\quad+e^{-np_k}\exp(-itp_k\sqrt{n})\exp(itrn^{-1/2})\left[-\frac{isg(n)n^{r-1}p_k^r}{(r-1)!}-\frac{is(r-1)g(n)n^{r-1}p_k^r}{(r-1)!(n-r+1)}-\frac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2}\right.\\
&\quad\left.+O\left(\frac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right)\right]\\
&= e^{-np_k}e^{-itp_k\sqrt{n}}e^{it(r-1)n^{-1/2}}\left\{\frac{isn^{r-1}g(n)p_k^r}{(r-1)!}-\frac{s^2n^{r-1}g^2(n)p_k^{r+1}}{2!(r-1)!}-O\left(s^3n^{r-1}g^3(n)p_k^{r+2}\right)\right.\\
&\quad+\left(1+\frac{it}{\sqrt{n}}-\frac{t^2}{2!n}-O(\frac{it^3}{3!n^{3/2}})\right)\left[-\frac{isg(n)n^{r-1}p_k^r}{(r-1)!}-\frac{is(r-1)g(n)n^{r-1}p_k^r}{(r-1)!(n-r+1)}-\frac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2}\right.\\
&\quad\left.\left.+O\left(\frac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right)\right]\right\}\\
&= e^{-np_k}e^{-itp_k\sqrt{n}}e^{it(r-1)n^{-1/2}}\left\{-\frac{is(r-1)g(n)n^{r-1}p_k^r}{(r-1)!(n-r+1)}-\frac{s^2n^{r-1}g^2(n)p_k^{r+1}}{2!(r-1)!}-\frac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2}\right.\\
&\quad+O\left(\frac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right)-O\left(s^3n^{r-1}g^3(n)p_k^{r+2}\right)\\
&\quad+\left(\frac{it}{\sqrt{n}}-\frac{t^2}{2!n}-O(\frac{it^3}{3!n^{3/2}})\right)\left[-\frac{isg(n)n^{r-1}p_k^r}{(r-1)!}-\frac{is(r-1)g(n)n^{r-1}p_k^r}{(r-1)!(n-r+1)}-\frac{s^2r^2g^2(n)n^rp_k^r}{2!r!(n-r+1)^2}\right.\\
&\quad\left.\left.+O\left(\frac{s^3g^3(n)n^rp_k^r}{(n-r+1)^3}\right)\right]\right\}.
\end{aligned}
\tag{6.1}
$$

Noting the uniform convergence of $e^{-itp_k\sqrt{n}}e^{it(r-1)n^{-1/2}}\to 1$ and Condition 2.1, it can be checked that all terms in (6.1) vanish under $\lim\sum_I$, except possibly the first three terms within

the curly brackets, *i.e.*,

$$\lim \sum_{k\in I} E_k = \lim \sum_{k\in I} \left\{ e^{-np_k} \left[-\frac{is(r-1)g(n)n^{r-1}p_k^r}{(r-1)!(n-r+1)} - \frac{s^2 n^{r-1} g^2(n) p_k^{r+1}}{2!(r-1)!} - \frac{s^2 r^2 g^2(n) n^r p_k^r}{2!r!(n-r+1)^2} \right] \right\}$$

$$= -\frac{is(r-1)}{(r-1)!} \lim \sum_{k\in I} \frac{g(n)n^{r-1}p_k^r}{(n-r+1)} e^{-np_k} - \frac{s^2}{2!(r-1)!} \lim \sum_{k\in I} n^{r-1} g^2(n) p_k^{r+1} e^{-np_k}$$

$$-\frac{s^2 r^2}{2!r!} \lim \sum_{k\in I} \frac{g^2(n) n^r p_k^r}{(n-r+1)^2} e^{-np_k}.$$

Condition 2.1 guarantees the existence of the second and the third terms above, and the existence of the third term implies that the first term is zero. Therefore 2) is checked and

$$\lim \sum_{k\in I} E_k = -\frac{s^2}{2(r-1)!} \left[\lim \sum_{k\in I} n^{r-1} g^2(n) p_k^{r+1} e^{-np_k} + r \lim \sum_{k\in I} \frac{g^2(n) n^r p_k^r}{(n-r+1)^2} e^{-np_k} \right]. \tag{6.2}$$

The convergence of $\sum_I E_k$ and hence of $\sum_I |E_k|$ guarantees (6).

For (5), since $B_k = \exp\left(-\frac{t^2}{2}p_k + O(t^3 p_k n^{-1/2})\right)$ and $-\frac{t^2}{2}p_k + O(t^3 p_k n^{-1/2}) \to 0$ uniformly,

$$|B_k - 1| \le \frac{|-\frac{t^2}{2}p_k + O(t^3 p_k n^{-1/2})|}{1 - |-\frac{t^2}{2}p_k + O(t^3 p_k n^{-1/2})|} \le O\left(\frac{t^2}{2}p_k + t^3 p_k n^{-1/2}\right)$$

and hence

$$\sum_I |B_k - 1| \le O\left(\frac{t^2}{2}\sum_I p_k) + \frac{|t^3|}{\sqrt{n}} \sum_I p_k\right) < O(t^2 + |t^3|).$$

□

6.2 Proof of Lemma 3.1.

Consider the partition of $K = I \cup II$. Since pe^{-np} has a negative derivative with respect to p on interval $(1/n, 1]$ and hence on $(r/n^{1-\delta^*}, 1]$ for large n, pe^{-np} attains its maximum at $p = r/n^{1-\delta^*}$. Therefore noting that there are at most n^{δ^*}/r indices in II,

$$0 \le \frac{g^2(n)}{n^2}\binom{n}{r} \sum_{II} p_k^r (1-p_k)^{n-r} \le \frac{g^2(n)}{n^2}\binom{n}{r} \sum_{II} p_k^r e^{-(n-r)p_k} \le \frac{g^2(n)}{n^2}\binom{n}{r} e^r \sum_{II} p_k e^{-np_k}$$

$$\le \frac{g^2(n)}{n^2}\binom{n}{r} e^r \sum_{II} \left(\frac{r}{n^{1-\delta^*}} e^{-\frac{nr}{n^{1-\delta^*}}}\right) \le \frac{g^2(n)}{n^2}\binom{n}{r} e^r \frac{n^{\delta^*}}{r} \left(\frac{r}{n^{1-\delta^*}} e^{-\frac{nr}{n^{1-\delta^*}}}\right)$$

$$= \frac{g^2(n)}{n^2}\binom{n}{r} e^r \frac{n^{\delta^*}}{n^{1-\delta^*}} e^{-n^{\delta^*}} \to 0.$$

Thus

$$\lim \frac{g^2(n)}{n^2} E(N_r) = \lim \frac{g^2(n)}{n^2}\binom{n}{r} \sum_I p_k^r (1-p_k)^{n-r} \tag{6.3}$$

and

$$\lim n^{r-2}g^2(n)\sum p_k^r \exp(-np_k) = \lim n^{r-2}g^2(n)\sum_I p_k^r \exp(-np_k). \tag{6.4}$$

On the other hand,

$$\tfrac{g^2(n)}{n^2}\tbinom{n}{r}\textstyle\sum_I p_k^r(1-p_k)^{n-r} \le \tfrac{g^2(n)}{n^2}\tbinom{n}{r}\sum_I p_k e^{-(n-r)p_k} \le \tfrac{g^2(n)}{n^2}\tbinom{n}{r}\exp(r\sup_I p_k)\sum_I p_k e^{-np_k}.$$

Furthermore, applying 2) and 3) of Lemma 2.6 in the first and the third steps below respectively leads to

$$\begin{aligned}\tfrac{g^2(n)}{n^2}\tbinom{n}{r}\textstyle\sum_I p_k^r(1-p_k)^{n-r} &\ge \tfrac{g^2(n)}{n^2}\tbinom{n}{r}\sum_I p_k^r \exp\left(-\tfrac{(n-r)p_k}{1-p_k}\right)\\ &\ge \tfrac{g^2(n)}{n^2}\tbinom{n}{r}\textstyle\sum_I p_k^r \exp\left(-\tfrac{np_k}{1-\sup_I p_k}\right) \ge \tfrac{g^2(n)}{n^2}\tbinom{n}{r}\sum_I \exp(-2n(\sup_I p_k)^2)p_k^r e^{-np_k}.\end{aligned}$$

Noting the fact that $\lim \exp(r\sup_I p_k) = 1$ and $\lim \exp(-2n(\sup_I p_k)^2) = 1$ uniformly by the definition of I,

$$\lim \frac{g^2(n)}{n^2}\binom{n}{r}\sum_I p_k^r(1-p_k)^{n-r} = \lim \frac{g^2(n)}{n^2}\binom{n}{r}\sum_I p_k^r e^{-np_k},$$

and hence, by (6.3) and (6.4) and by the fact that $\binom{n}{r} \sim n^r/r!$, the equivalence of the first parts of Condition 2.1 and Condition 3.1 is established:

$$\lim \frac{g^2(n)}{n^2}E(N_1) = (1/r!)\lim n^{r-2}g^2(n)\sum p_k^r \exp(-np_k).$$

The equivalence of the second parts can be established similarly. □

6.3 Proof of Theorem 3.3.

Based on Theorem 3.2, it suffices to show that the variances of

$$\hat{c}_r = \frac{r!g^2(n)}{n^2}N_r \qquad \textit{and} \qquad \hat{c}_{r+1} = \frac{(r+1)!g^2(n)}{n^2}N_{r+1}$$

approach zero as n increases to infinity.

$$Var(\hat{c}_r) = \frac{(r!)^2g^4(n)}{n^4}Var(N_r) = \frac{(r!)^2g^4(n)}{n^4}\left\{E(N_r^2) - [E(N_r)]^2\right\}. \tag{6.5}$$

$$E(N_r^2) = E(N_r) + \textstyle\sum_{k\neq j} \frac{n!}{r!r!(n-2r)!} p_k^r p_j^r (1-p_k-p_j)^{n-2r}$$

$$(EN_r)^2 = \left[\tbinom{n}{r} \textstyle\sum p_k^r (1-p_k)^{n-r}\right]^2$$

$$= \tbinom{n}{r}^2 \textstyle\sum_{k\neq j} p_k^r p_j^r (1-p_k)^{n-r}(1-p_j)^{n-r} + \tbinom{n}{r}^2 \textstyle\sum_k p_k^{2r}(1-p_k)^{2n-2r}.$$

By the first part of Condition 3.1, $\frac{(r!)^2 g^4(n)}{n^4} E(N_r) \to 0$ since $g^2/n^2 \to 0$.

Therefore

$$\lim \tfrac{(r!)^2 g^4(n)}{n^4} \left[E(N_r^2) - (EN_r)^2\right]$$

$$\leq \lim \tfrac{g^4(n)}{n^4} \left[\textstyle\sum_{k\neq j} \frac{n!}{(n-2r)!} p_k^r p_j^r (1-p_k-p_j)^{n-2r} - \frac{(n!)^2}{[(n-r)!]^2} \sum_{k\neq j} p_k^r p_j^r (1-p_k)^{n-r}(1-p_j)^{n-r}\right]$$

$$= \lim \tfrac{g^4(n)}{n^4} \left[\textstyle\sum_{k\neq j} \frac{n!}{(n-2r)!} p_k^r p_j^r (1-p_k-p_j)^{n-2r} - \frac{n!}{(n-2r)!} \sum_{k\neq j} p_k^r p_j^r (1-p_k)^{n-r}(1-p_j)^{n-r}\right]$$

$$+ \lim \tfrac{g^4(n)}{n^4} \left[\tfrac{n!}{(n-2r)!} - \tfrac{(n!)^2}{[(n-r)!]^2}\right] \left[\textstyle\sum_{k\neq j} p_k^r p_j^r (1-p_k)^{n-r}(1-p_j)^{n-r}\right].$$

The second term above is bounded by

$$\lim \tfrac{g^4(n)}{n^4} \left[\tfrac{n!}{(n-2r)!} - \tfrac{(n!)^2}{[(n-r)!]^2}\right] \left[\textstyle\sum_k \sum_j p_k^r p_j^r (1-p_k)^{n-r}(1-p_j)^{n-r}\right]$$

$$= \lim \left[\tfrac{n!}{(n-2r)!} - \tfrac{(n!)^2}{[(n-r)!]^2}\right] \tbinom{n}{r}^{-2} \left[\tfrac{g^2(n)}{n^2} \tbinom{n}{r}^2 \textstyle\sum_k p_k^r (1-p_k)^{n-r}\right]^2$$

$$= \lim \left[\tfrac{n!}{(n-2r)!} - \tfrac{(n!)^2}{[(n-r)!]^2}\right] \tbinom{n}{r}^{-2} \left[\tfrac{g^2(n)}{n^2} E(N_r)\right]^2$$

$$= \left(\tfrac{c_r}{r!}\right)^2 \lim \left[\tfrac{n!}{(n-2r)!} - \tfrac{(n!)^2}{[(n-r)!]^2}\right] \tbinom{n}{r}^{-2} = 0.$$

Noting $(1-p_j-p_k)^{n-2r} \leq (1-p_j-p_k+p_jp_k)^{n-2r} = [(1-p_j)(1-p_k)]^{n-2r}$, and therefore

$$\lim \tfrac{(r!)^2 g^4(n)}{n^4} \left[E(N_r^2) - (EN_r)^2\right]$$

$$\leq \lim \tfrac{g^4(n)}{n^4} \tfrac{n!}{(n-2r)!} \left[\textstyle\sum_{k\neq j} p_k^r p_j^r [(1-p_j)(1-p_k)]^{n-2r} - \sum_{k\neq j} p_k^r p_j^r (1-p_k)^{n-r}(1-p_j)^{n-r}\right]$$

$$= \lim \tfrac{g^4(n)}{n^4} \tfrac{n!}{(n-2r)!} \left\{\textstyle\sum_{k\neq j} p_k^r p_j^r [(1-p_j)(1-p_k)]^{n-2r} \left[1-(1-p_k)^r(1-p_j)^r\right]\right\}$$

$$\leq \lim \tfrac{g^4(n)}{n^4} \tfrac{n!}{(n-2r)!} \left\{\textstyle\sum_{k\neq j} p_k^r p_j^r [(1-p_j)(1-p_k)]^{n-2r} \left\{1-[1-(p_k+p_j)]^r\right\}\right\}.$$

Noting $1-(1-x)^r \leq (2^r-1)x$ for all $x \in [0,1]$,

$$\lim \tfrac{(r!)^2 g^4(n)}{n^4} \left[E(N_r^2) - (EN_r)^2\right]$$

$$\leq \lim \tfrac{g^4(n)}{n^4} \tfrac{n!(2^r-1)}{(n-2r)!} \left\{\textstyle\sum_{k\neq j} p_k^r p_j^r [(1-p_j)(1-p_k)]^{n-2r}(p_k+p_j)\right\}$$

$$= 2\lim \tfrac{g^4(n)}{n^4} \tfrac{n!(2^r-1)}{(n-2r)!} \left\{\textstyle\sum_{k\neq j} p_k^{r+1} p_j^r [(1-p_j)(1-p_k)]^{n-2r}\right\}.$$

On the other hand,

$$\begin{aligned}
&\textstyle\sum_{k\neq j} p_k^{r+1}p_j^r(1-p_k)^{n-2r}(1-p_j)^{n-2r}\\
&\quad= \left(\textstyle\sum_{k\neq j,p_k\le p_j}+\sum_{k\neq j,p_k>p_j}\right) p_k^{r+1}p_j^r(1-p_k)^{n-2r}(1-p_j)^{n-2r}\\
&\quad\le \textstyle\sum_{k\neq j,p_k\le p_j} p_k^r p_j^{r+1}(1-p_k)^{n-r}(1-p_j)^{n-3r}\\
&\qquad+\textstyle\sum_{k\neq j,p_k>p_j} p_k^{r+1}p_j^r(1-p_k)^{n-3r}(1-p_j)^{n-r}\\
&\quad\le 2\textstyle\sum_k\sum_j p_k^r p_j^{r+1}(1-p_k)^{n-r}(1-p_j)^{n-3r}\\
&\quad\le 2\textstyle\sum_k p_k^r(1-p_k)^{n-r}\sum_j p_j^{r+1}(1-p_j)^{n-3r} = 2\binom{n}{r}^{-1}E(N_r)\textstyle\sum_j p_j^{r+1}(1-p_j)^{n-3r}.
\end{aligned}$$

Noting that $p^r(1-p)^{n-3r}$ attains its maximum at $p=r/(n-2r)$ and hence $p^r(1-p)^{n-3r}\le r^r/(n-2r)^r$,

$$\textstyle\sum_{k\neq j} p_k^{r+1}p_j^r(1-p_k)^{n-2r}(1-p_j)^{n-2r}\le 2r^r\binom{n}{r}^{-1}(n-2r)^{-r}E(N_r).$$

Finally

$$\begin{aligned}
\lim\tfrac{(r!)^2g^4(n)}{n^4}\left[E(N_r^2)-(EN_r)^2\right] &\le 4\lim\tfrac{g^4(n)}{n^4}\tfrac{n!(2^r-1)}{(n-2r)!}\tfrac{r^r}{\binom{n}{r}(n-2r)^r}E(N_r)\\
&= 4r^r(2^r-1)c_r\lim\left[\tfrac{g^2(n)}{n^2}\tfrac{(n-r)!}{(n-2r)!(n-2r)^r}\right]=0.
\end{aligned}$$

The consistency of $\hat{c}_r$ follows. The consistency of $\hat{c}_{r+1}$ can also be similarly proved. □

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