3 Solution of Homework

Problem 3.1. Prove from the axioms of incidence (I.1) (I.2) (I.3) that there exist two different lines through every point.

Answer. By axiom (I.3b), there exist at least three points that do not lie on a line. We call them $A, B$ and $C$. Let any point $P$ be given. In the case that point $P$ is different from all three points $A, B, C$, we draw the three lines $PA, PB$ and $PC$. At least two of them are different since $A, B, C$ do not lie on a line. In the case that point $P$ is one of the three points $A, B, C$, we draw the three lines $AB, BC$ and $CA$. These are three different lines, and two of them go through the given point $P$. In both cases we have obtained two different lines through the arbitrary point $P$.

Problem 3.2. As far as two-dimensional geometry is concerned, Hilbert’s Proposition 1 reduces to one simple statement: any two different lines either intersect in one point, or are parallel. Give two or more further useful formulations of this statement.

Answer. Here are three possible answers:

- Any two different lines which are not parallel, have a unique point of intersection.
- If two lines have two or more points in common, they are equal.
- Two different lines have at most one point in common.

Lemma 1 (Proclus’ Lemma). In any affine plane,

- a third line intersecting one of two parallel lines intersects the other one, too.
- a third line parallel to one of two parallel lines, is parallel to the other one, too.

Problem 3.3. Explain why Proclus’ Lemma is an easy consequence of the uniqueness of parallels. Convince yourself that, conversely, Proclus’ Lemma implies the uniqueness of parallels.
Answer. Suppose towards a contradiction that the transversal \( t \) intersects one of the parallel lines \( l \) and \( m \), but not the other one.

We may assume that \( P \) is the intersection point of lines \( t \) and \( m \). If lines \( t \) and \( l \) would not intersect, then \( t \) and \( m \) would be two different parallels of line \( l \) through point \( P \). This contradicts the uniqueness of parallels.

Conversely, we now assume Proclus’ Lemma to be true and check the uniqueness of the parallel to a given line \( l \) through a given point \( P \) not on line \( l \). Let \( m \) and \( t \) be two parallels to line \( l \) through point \( P \)—these may be equal or different lines. The line \( t \) is a transversal intersecting one of the two parallel lines \( m \parallel l \) at point \( P \). Hence it intersects the second line \( l \), too, contrary to the assumption. The only possibility left is that \( m = t \). Hence the parallel to a given line through a given point is unique.

\[ \text{Problem 3.4 (The six-point incidence geometries).} \quad \text{Find all non-isomorphic incidence geometries with six points. Count how many non-isomorphic models do exist.}\]

\text{For the models different from the handshake and straight fan, mark all three-point lines with different blue shades, and any four-point line in red.}\n
\text{Describe the properties of their points and lines. Count the lines. Which parallel property (elliptic, Euclidean, hyperbolic, or neither) does hold?}\n
\text{Answer. There are nine non-isomorphic six-point incidence planes:}\n
1. The handshake model with 15 short lines.
2. The model with only one three-point line has 13 lines.
3. The model with only one four-point line has \( 1 + 1 + 2 \cdot 4 = 10 \) lines.
4. The model with two parallel three-point lines has \( 2 + 3 \cdot 3 = 11 \) lines.
5. The model with two intersecting three-point lines has \( 2 + 2 \cdot 2 + 5 = 11 \) lines.
6. The model with a intersecting three-point line and four-point line has \( 2 + 2 \cdot 3 = 8 \) lines.
7. The model with three intersecting three-point lines forming a ”triangle” has \( 3 + 3 \cdot 2 = 9 \) lines.
8. The model with three intersecting three-point lines forming a ”triangle” and the in-circle a fourth three-point line has \( 4 + 3 = 7 \) lines.
9. The straight fan with \( 1 + 5 = 6 \) lines.

Only the straight fan is elliptic. There is no Euclidean model. There are several mixed models, with are not hyperbolic neither.

\[ \text{Problem 3.5 (A nine point incidence geometry).} \quad \text{Find a model of incidence geometry with nine points, which satisfies}\]
"(I3+) "Every line contains exactly three points." 
and for which the Euclidean parallel axiom does hold:

"For every line \( l \) and every point \( P \) not on \( l \), there exists exactly one parallel to the line \( l \) through \( P \)."

It is enough to provide a drawing to explain the model. How many lines does the model have?

Use colors for the lines. Choose clearly different colors for intersecting lines, but give each set of three parallel lines different shades of nearby color.

![Figure 1: A nine-point incidence geometry](image)

Answer. There is exactly one such model. It has 12 lines with four subsets of lines, each one consists of three parallel lines. Through each point go four lines which have all four possible slopes 0, 1, −1 and vertical.
Problem 3.6. Give a highly symmetric illustration for the Fano plane based on an equilateral triangle. Your symmetric illustration is really isomorphic to the projective plane from page 4 above, which was obtained above by completion. (See the lecture or the online lecture notes)

Denote the seven points with the same names in both drawings, consistently in a way to show the isomorphism. After you have obtained the isomorphism, color the lines with seven different colors. Give the corresponding lines in the other model the same colors.

Answer. The figure on page 5 give an illustration based on an equilateral triangle. To check that this symmetric illustration is isomorphic to the illustration I have given on the left side, one needs to names the points in both illustrations in a way that the incidence relations hold for the same names. Thus the isomorphism is given by the correspondence of names.

To find such an isomorphism, the key observation is that a triangle can be mapped to any triangle, but afterwards the correspondence of the remaining points is uniquely determined.

Problem 3.7 (The rational plane is not Euclidean). The rational Cartesian plane $\mathbb{Q}^2$ is an incidence plane, but not a Hilbert plane. Which are the two among Hilbert’s axioms of incidence (I.1)(I.2)(I.3), and the axioms of order, congruence, parallelism, and continuity that do not hold in this case?

We may assume to be known that $\sqrt{2}$ is not rational. Explain with a simple counterexample involving $\sqrt{2}$ that one of the axioms of congruence does not hold.

Answer. Only the axiom of congruence (III.1), and the completeness axiom (V.2) are not satisfied in the rational Cartesian plane $\mathbb{Q}^2$.

To see that the axiom of congruence (III.1) is violated, we take the unit square with vertices $(0,0), (1,0), (0,1), (1,1)$ and try to transfer its diagonal onto the positive $x$-axis,
Figure 3: The symmetric drawing of the Fano-plane is really isomorphic to the projective completion of the affine plane of order 2.

the ray $(0,0)(1,0)$. This is impossible because a segment $(0,0)(\frac{p}{q},0)$ with length $\sqrt{2}$ does not exist.

The completeness axiom (V.2) is violated, since it is possible to adjoin additional points into the rational plane. Any point $(x,y)$ with one or both coordinates irrational will do.

Problem 3.8. Explain why unique angle transfer according to axiom (III.4) is possible in the rational plane $\mathbb{Q}^2$, and indeed axiom (III.4) does hold. You may assume the addition theorem for the tangent function to be known:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

For a ray $\overrightarrow{AB}$ we define the slope

$$m = \frac{b_2 - a_2}{b_1 - a_1}$$

with $A = (a_1, a_2)$ and $B = (b_1, b_2)$. This way, opposite rays get the same slope. Hence we need to restrict ourselves at first to the rays pointing into one half plane. I choose the right half plane for this purpose and define: A ray $\overrightarrow{AB}$ is going towards right if and only if the inequality $b_1 > a_1$ holds for the first coordinates of $A$ and $B$. 

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Next we get a notion of congruence of angles. I restrict myself at first to angles both sides of which are going towards the right. Let \( \angle(h, k) \) be an angle towards right, and let \( m_h \) and \( m_k \) be the slopes of its two sides. We define the tangent of the angle by

\[
\tan \angle(h, k) = \frac{|m_h - m_k|}{1 + m_h m_k}
\]

as an element of \( \mathbb{F} \cup \{\infty\} \). The value infinity occurs exactly for a right angle because of the denominator \( 1 + m_h m_k = 0 \).

**Question.** Convince yourself that for an acute angle \( \tan \angle(h, k) > 0 \) is finite, for an obtuse angle \( \tan \angle(h, k) < 0 \) is finite. Note the absolute value in the numerator!

We define the congruence of any two angles \( \angle(h, k) \) and \( \angle(h', k') \), both going towards right, by postulating

\[
\angle(h, k) \cong \angle(h', k') \iff \tan \angle(h, k) = \tan \angle(h', k')
\]

It is rather straightforward but a bid tedious to extend the notion of congruence to arbitrary angles, by means of supplementary and vertical angles.

**Indication of the solution.** Given is an angle \( \angle(h, k) \) and a ray \( h' \). For simplicity, I assume that all three rays \( h, k, h' \) go to the right and \( m_h > m_k \). The tangent of the given angle is

\[
\tan \angle(h, k) = \frac{m_h - m_k}{1 + m_h m_k}
\]

The slope of the second side \( h' \) or \( h'' \) of the angle transferred to the requested half plane is to be obtained. This can be done simply with the addition theorem for tangents, provided that the new ray \( h' \) points towards the right, too. In that case

\[
m'_h = \frac{m'_h + \tan \angle(h, k)}{1 - m'_k \tan \angle(h, k)}
\]

is its slope. One checks that

\[
\frac{m'_h - m'_k}{1 + m'_h m'_k} = \tan \angle(h, k)
\]

We see that we get the correct transfers provided \( 1 + m'_h m'_k \) is positive or negative as needed to result in \( \tan \angle(h', k') = \tan \angle(h, k) \). If we get \( \tan \angle(h', k') = -\tan \angle(h, k) \), we need to replace the ray \( h' \) by its opposite ray; the formula for the slope \( m'_h \) is still correct.

The transfer to the other half-plane leads to the ray \( h'' \) with slope

\[
m''_h = \frac{m'_k - \tan \angle(h, k)}{1 + m'_k \tan \angle(h, k)}
\]
One checks that
\[ \frac{m''_h - m'_k}{1 + m''_hm'_k} = -\tan \angle(h, k) \]
We see that we get the correct transfers provided \(1 + m'_h m'_k\) is positive or negative as needed to yield the requested equation \(\tan \angle(h'', k') = \tan \angle(h, k)\). If we obtain \(\tan \angle(h'', k') = -\tan \angle(h, k)\) instead, we need to replace the ray \(h''\) by its opposite ray in order to transfer the angle correctly. Well, I see clearly enough that all remaining details can be worked out. We thus have checked that the requested rays \(h'\) and \(h''\) have rational slope and hence exist in the rational plane \(\mathbb{Q}^2\).

\[\boxed{10}\] Problem 3.9 (Three points on a line). In a projective coordinate plane three points with the homogeneous coordinates \((x_1, y_1, z_1), (x_2, y_2, z_2)\) and \((x_3, y_3, z_3)\) are given. Check that the three points lie on a line if and only if the determinant

\[
\begin{vmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3 \\
  z_1 & z_2 & z_3
\end{vmatrix} = 0
\]

Simple solution. The three points of the projective plane with the homogeneous coordinates \((x_1, y_1, z_1), (x_2, y_2, z_2)\) and \((x_3, y_3, z_3)\) lie on a line if and only if the three lines through the origin

\[
\{(\lambda x_1, \lambda y_1, \lambda z_1) : \lambda \in \mathbb{F}\} \quad \{(\lambda x_2, \lambda y_2, \lambda z_2) : \lambda \in \mathbb{F}\} \quad \{(\lambda x_3, \lambda y_3, \lambda z_3) : \lambda \in \mathbb{F}\}
\]
lie in a plane in \(\mathbb{F}^3\). It is known that this happens if and only if the determinant of their coordinates from above is zero. \(\Box\)