12 Inversion by a Circle

12.1 Definition and construction of the inverted point

Let $D$ be a open circular disk of radius $R$ and center $O$, and denote its boundary circle by $\partial D$.

Definition 12.1 (Inversion by a circle). The inversion by the circle $\partial D$ is defined to be the mapping from the plane plus one point $\infty$ at infinity to itself, which maps an arbitrary point $P \neq O$, to its inverse point $P'$—defined to be the point on the ray $\overrightarrow{OP}$ such that $|OP| \cdot |OP'| = R^2$. Hence, especially, all the points of $\partial D$ are mapped to themselves. The inversion maps the origin $O$ to $\infty$, and $\infty$ to $O$. We denote the images by inversion with primes.

Problem 12.1. Do an example for the construction of the inverse point. Use the theorems related to the Pythagorean theorem.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{construction_inverse_point}
\caption{Construction of the inverse point}
\end{figure}

Construction 12.1 (Inversion of a given point). Let $P$ be the given point. One erects the perpendicular onto ray $\overrightarrow{OP}$ at point $P$. Let $C$ be an intersection point of the perpendicular with $\partial D$. Next one erects the perpendicular on radius $OC$ at point $C$, and gets a tangent to circle $\partial D$. The inverse point $P'$ is the intersection of that tangent with the ray $OP$.

Reason. Indeed, by the leg theorem, $|OP| \cdot |OP'| = |OC|^2 = R^2$.

Remark. Here is an alternative justification of the construction. Put in Thales’ circle with diameter $CP'$. By the converse Thales’ theorem, $P$ lies on that circle. Then use the chord-tangent theorem Euclid III.36. One concludes $|OP| \cdot |OP'| = |OC|^2 = R^2$, once more.
12.2 The gear of Peaucollier

The gear of Peaucellier allows a construction of the inverse point.

**Problem 12.2** (The gear of Peaucellier). Six stiff rods are linked and can be turned flexibly by each other within a plane. Two rods of length $a$ are linked to each other at point $O$, the remaining endpoints are $A$ and $C$. The four remaining rods have a different length $b$—they are linked to a rhombus, and the points $A$ and $C$ on one diagonal are linked to the first two rods.

(i) Assume that $a > b$. Prove that the endpoints of the other diagonal of the rhombus are inverse points by a suitable circle $\partial D$ with center $O$. Find the radius of this circle. You can use the Theorem of chords Euclid III.35 for a second circle $\mathcal{L}$ with center $A$.

(ii) Check that the assumption $a > b$ implies that point $O$ lies outside circle $\mathcal{L}$. Let $OT$ be a tangent to the same circle. Use Euclid III.36 to get the length of $OT$.

(iii) Check the Theorem of Pythagoras for the right triangle $\triangle OTA$.

(iv) Prove that the circles $\partial D$ and $\mathcal{L}$ intersect each other perpendicularly.

(v) What happens in the case $a < b$?
Answer. We draw the circle $\mathcal{L}$ around $A$ through points $P$ and $P'$. This circle intersects line $OA$ in two endpoints $E$ and $F$ of a diameter. Too, the three points $O, P$ and $P'$ lie on a line.

(i) The assumption $a > b$ implies that $P$ and $P'$ lie on the same side of $O$. We use the Theorem of chords Euclid III.35 (see 5.3) to conclude

$$|OP| \cdot |OP'| = |OE| \cdot |OF| = (a + b)(a - b)$$

Hence $P$ and $P'$ are inverse points by a circle of radius $R$, and

$$R^2 = (a + b)(a - b) = a^2 - b^2$$

(ii) By the Theorem of chord and tangent Euclid III.36 (see 5.3), the length of the tangent from point $O$ to the circle $\mathcal{L}$ satisfies

$$|OT|^2 = |OP| \cdot |OP'|$$

Together, we get $|OT|^2 = a^2 - b^2 = R^2$, and hence $|OT| = R$.

(iii) The right triangle $\triangle OTA$ has legs $|OT| = R, |TA| = b$, and hypotenuse $|OA| = a$. Hence the Theorem of Pythagoras tells that $a^2 = R^2 + b^2$, as we have already seen above.
(iv) The segment $AT$ is a radius of circle $\mathcal{L}$. The perpendicular segment $OT$ is a radius of circle $\partial D$, and at the same time a tangent of circle $\mathcal{L}$. Hence $AT$ is a tangent of circle $\partial D$, and the two circles intersect perpendicularly.

(v) In the case $b > a$, the point $O$ lies between $P$ and $P'$. Now $P$ and $P'$ are antipodal points by a circle $\partial D$ of radius $R_a$, but $R_a^2 = b^2 - a^2$. Point $O$ lies now inside this circle. The drawing on page 577 provides an example.

**Figure 12.4:** In case $b > a$, the gear of Peaucollier constructs the antipodal point.

### 12.3 Invariance properties of inversion

After two further definitions, we can state the main result of this section.

**Definition 12.2.** A generalized circle is defined to be either a circle or a straight line.

**Definition 12.3.** The cross ratio of four points $A, B, C, D$ is defined as

$$ (AC, BD) = \frac{|AB| \cdot |CD|}{|CB| \cdot |AD|} $$

**Remark.** Remember:

$$ A \rightarrow C \quad B \rightarrow D $$

$$ C \leftarrow A \quad B \rightarrow D $$
Main Theorem 22. The inversion by a circle maps generalized circles to generalized circles, conserves angles, and conserves the cross ratio.

Definition 12.4. The power of a point $O$ with respect to a circle $C$ is defined by

$$p = |OA| \cdot |OB|$$

Here $A$ and $B$ are the two intersection points of any line $l$ through $O$ with the circle $C$. The power is negative for points inside the circle—and positive for points outside the circle.

Soundness of definition. Let $k$ be any other line intersecting the circle $C$, now in the points $P$ and $Q$. We need to confirm that we get the same value for the power, using line $k$. Indeed $|OA| \cdot |OB| = |OP| \cdot |OQ|$ by Euclid III.35 and III.36. Hence

$$p = |OP| \cdot |OQ|$$

which shows that $p$ does not depend on the choice of the line. Therefore the power is well defined.

Proposition 12.1. The inversion by a circle maps generalized circles to generalized circles.

Figure 12.5: A circle, not going through $O$ is mapped to a circle.
Circles not through $O$ are mapped to circles. Given is a circle $C$ which does not go through center $O$. We prove that its inverse image is a circle, too.

Take any two lines $l$ and $k$ through $O$. Let $A, B$ and $P, Q$ be their intersection points with $C$. By definition of inversion, $|OA| \cdot |OA'| = |OB| \cdot |OB'| = R^2$. And hence

$$|OA'| \cdot |OB'| = \frac{R^4}{|OA| \cdot |OB|} = \frac{R^4}{p}$$

where $p$ is the power of point $O$ relative to circle $C$. For the points $P$ and $Q$ on the second line $k$, one calculates again

$$|OP'| \cdot |OQ'| = \frac{R^4}{|OP| \cdot |OQ|} = \frac{R^4}{p}$$

Since this is the same value $|OP'| \cdot |OQ'| = |OA'| \cdot |OB'|$

Euclid III.35 and III.36 imply that the four points $A', B', P'$ and $Q'$ lie on a circle $C'$. Indeed, since the line $k$ is arbitrary, we have shown that the images of all points of $C$ lie on that circle $C'$.

**Question.** What is the power of point $O$ relative to the inverted circle $C'$.
**Answer.** The calculation above show that \( \frac{R^2}{p} \) is the power of point \( O \) relative to the inverted circle \( C' \).

In the case that point \( O \) lies outside of circle \( C \), one can conclude even more. In the limiting case that \( P \) moves to \( Q \), line \( k \) becomes a tangent from point \( O \) to circle \( C \). In the same process, \( P' \) moves to \( Q' \). Hence the ray \( OP = OP' \) becomes a common tangent of the two circles \( C \) and \( C' \). Hence both common tangents of circles \( C \) and \( C' \) intersect at point \( O \).

This construction suggests that circles \( C \) and \( C' \) are preimage and image for a central dilation with center \( O \). We now prove this claim in both cases that \( O \) lies outside or inside of \( C \).

**Lemma 12.1.** A central dilation \( z \) with center \( O \) and ratio 

\[
k = \frac{R^2}{p}
\]

maps the circle \( C \) to the circle \( C' \). The intersection points \( P \) and \( Q \) of circle \( C \) with any central ray \( k \) are mapped as 

\[
P \mapsto Q' \quad \text{and} \quad Q \mapsto P'
\]

—in a different way as inversion by circle \( \partial D \) maps them.

The central dilation \( z \) maps the center \( Z \) of circle \( C \) to the center \( Z_* \) of circle \( C' \), and the touching point \( C \) of a common tangent to touching point \( C_2 \). Hence 

\[
\frac{|OC_2|}{|OC|} = \frac{|OZ_*|}{|OZ|} = k
\]

**Proof.** We use proportions. Indeed 

\[
\frac{|OQ'|}{|OP|} = \frac{|OP'|}{|OQ|} = \frac{|OP'| \cdot |OP|}{|OQ| \cdot |OP|} = \frac{R^2}{p} = k
\]

Hence the dilation \( z \) maps \( P \mapsto Q' \) and \( Q \mapsto P' \). Too, the center \( Z \) of circle \( C \) is mapped to the center \( Z_* \) of circle \( C' \), and the touching point \( C \) of a common tangent to touching point \( C_2 \). \( \square \)

A *circle through \( O \) is mapped to a line.* We now consider the exceptional case that the point \( O \) lies on \( C \), and prove that its image by inversion is a line. Let \( OA \) be a diameter of circle \( C \) and \( P \) be an arbitrary point on that circle. We erect the perpendicular \( c_* \) on ray \( OA \) at the inverse point \( A' \). Let \( P_* \) be the intersection of \( c_* \) with the ray \( OP \). The right triangles \( \Delta OAP \) and \( \Delta OP_*A \) are similar. Hence by Euclid VI.6, corresponding sides have the same ratio: 

\[
\frac{|OP|}{|OA|} = \frac{|OA'|}{|OP_*|}
\]

and hence \( |OP| \cdot |OP_*| = |OA| \cdot |OA'| = R^2 \). Thus \( P_* = P' \) is the inverse image of point \( P \), and the line \( c_* = C' \) is the inverse image of \( C \). \( \square \)
Proposition 12.2. The inversion by a circle conserves angles.

The angle between a radial ray and a circle not through $O$ is conserved. To show angles are conserved, we need to map further objects by the dilation $z$. Let $t$ be the tangent to circle $C$ at point $P$. The dilation $z$ maps the tangent $t$ to the tangent $t_*$ to the circle $C'$ at point $Q'$. By simple facts about central dilations, the tangents $t$ and $t_*$ are parallel. Hence by Euclid I.29, the lines $t$ and $t_*$ intersect the ray $\overrightarrow{OP}$ in congruent angles $\alpha = \alpha''$. Now let $t_2$ be the tangent to circle $C'$ at the point $P'$. (Why is $t_2$ not the image of $t$ under the inversion? This question distracts a bid, but see: the image of tangent $t$ is a circle though $O$, and has a common tangent with $C'$ at point $P'$.)

In general, the tangents $t_*$ and $t_2$ intersect, say at point $S$. We get an isosceles $\triangle SP'Q'$ with two congruent base angles

$$\alpha = \angle SQ'P' \cong \angle SP'Q'$$

Thus the tangent $t$ to circle $C$ at $P$, and the tangent $t_2$ to circle $C'$ at $P'$ both intersect the projection ray $\overrightarrow{OP}$ at angle $\alpha$.

In the exceptional case that the tangents $t_*$ and $t_2$ are parallel, they both intersect the ray $\overrightarrow{OP}$ at right angles. Since the tangents $t_*$ and $t$ are always parallel, both the tangent $t$ to circle $C$ at $P$, and the tangent $t_2$ to circle $C'$ at $P'$ intersect the projection ray $\overrightarrow{OP}$ at right angle. \qed
Problem 12.3. Show that the angle between a central ray and a circle through $O$ is conserved by inversion. To this end, prove that the three angles $\alpha, \alpha'$ and $\alpha''$ in the figure on page 581 are congruent.

Answer.

The angle between generalized circles is conserved. It is now easy to see that angles between circles are conserved. One maps two circles $C_1, C_2$ intersecting at point $P$ into two generalized circles $C_1', C_2'$. They intersect at point $P'$. One puts in the common projection ray $\overrightarrow{OP}$ and uses angle addition. The angle between the two generalized circles $C_1$ and $C_2$ is congruent to the angle between $C_1'$ and $C_2'$. It is left to the reader to check the remaining cases—involving generalized circles.

Proposition 12.3 (Conservation of the Cross Ratio). The inversion by a circle conserves the cross ratio of any four points $A, B, C, D$. (The four points need not lie on a circle.)

Reason. In figure on page 583, the circle through points $A, B, C$ is mapped by inversion to the circle through the inverted points $A', B', C'$. On the inverted circle $C'$, both the inverted points $A', B', C'$, and the dilated points $A_2, B_2, C_2$ are marked.

The first goal is to show that

\[
\frac{|A'B'|}{|C'B'|} = \frac{|AB|}{|CB|} \frac{|OC_2|}{|OA_2|}
\]
The easy part is to use the central dilation \( z \). Because central dilations conserve ratios, we get

\[
\frac{|AB|}{|CB|} = \frac{|A_2B_2|}{|C_2B_2|} \tag{1}
\]

We need now to relate the distances \(|A_2B_2|\) and \(|C_2B_2|\) to the distances \(|A'B'|\) and \(|C'B'|\).

To this end, we use angles in circle \( C' \) to find similar triangles

\[ \triangle OA_2B_2 \sim \triangle OB'A' \]

Clearly \( \triangle OA_2B_2 \) and \( \triangle OB'A' \) have a common angle at vertex \( O \). By Euclid III.35, we get supplementary \( \angle B'B_2A_2 \) and \( \angle B'A'A_2 \), because these angles subtend the two disjoint arcs from \( B' \) to \( A_2 \) on circle \( C' \). Because the angles \( \angle OB_2A_2 \) and \( \angle B'B_2A_2 \) are supplementary, we get congruent angles \( \angle OB_2A_2 \cong \angle B'A'A_2 \cong \angle B'A'O \). Hence the two triangles have two pairs of congruent angles. Because the angle sum in any triangle is two right angles (Euclid I.32), all three angles of the two triangles are pairwise congruent. Hence, by Euclid VI.4, these triangles are similar. Hence

\[
\frac{|A'B'|}{|OB'|} = \frac{|A_2B_2|}{|OA_2|}
\]

This ratio is actually \( \frac{\sin \alpha'}{\sin \omega} \). Similarly one gets that

\[
\frac{|C'B'|}{|OB'|} = \frac{|C_2B_2|}{|OC_2|}
\]

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**Figure 12.9:** Relation of the ratio of three points, and ratio of their inverted images.
and dividing the two ratios yields
\[
\frac{|A'B'|}{|C'B'|} = \frac{|A_2B_2|}{|C_2B_2|} \cdot \frac{|OC_2|}{|OA_2|}
\]

Now (1) and (2) imply the claim (*). Now one can use a similar relation for the three points \(A, D\) and \(C\), just replacing point \(B\) by point \(D\):
\[
((**)) \quad \frac{|A'D'|}{|C'D'|} = \frac{|AD|}{|CD|} \cdot \frac{|OC_2|}{|OA_2|}
\]

(It is not required that all four points \(A', B', C', D'\) lie on one circle, so that second relationship uses possibly a different pair of circles.) Division of the two relations (*) and (**) cancels out the last fraction \(\frac{|OC_2|}{|OA_2|}\) on the left, and leads to an equation between of cross ratios:
\[
(A'C', B'D') = (AC, BD)
\]