Problem 1 (Constructions in the Poincaré disk). Describe the following constructions for the Poincaré disk model, and provide a drawing for an adequate example.

(a) For a given point $P$, find the inverse image $P'$ and the polar $P^\perp$.

(b) For two given points $P$ and $Q$ not on a diameter, construct the hyperbolic line through $P$ and $Q$. 
Problem 2 (Constructions in the Poincaré disk). Describe the following constructions for the Poincaré disk model, and provide a drawing for an adequate example.

(c) For a given center \( P \neq O \) and a point \( R \) not on the diameter \( OP \), find the hyperbolic circle around \( P \) through point \( R \).

(d) For a given center \( P \neq O \) and a point \( R \) on the diameter \( OP \), find the hyperbolic circle around \( P \) through point \( R \).
Problem 3 (Constructions in the Klein disk). Describe the following constructions for the Klein disk model, and provide a drawing for an adequate example.

(a) For a given line $l$ and point $P$ not on $l$, drop the perpendicular from $P$ onto $l$.

(b) For a given line $l$ and point $P$ on $l$, erect the perpendicular at $P$ onto $l$. 
Problem 4 (Constructions in the Klein disk). Describe the following constructions for the Klein disk model, and provide a drawing for an adequate example.

(c) For a given line $l$ and point $P$ not on $l$, construct the image $P'$ obtained by reflection across $l$.

(d) For a given segment $PQ$ and another point $R$ on the line $PQ$, construct a congruent segment $RS \cong PQ$ on the same line.
Problem 5. Write down the automorphic collineation \( \phi_L \) corresponding by formula (1.1)

\[
x'_1 = \frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}} \quad \text{and} \quad x'_2 = \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}}
\]

to the standard Lorentz boost

\[
A := L = \begin{bmatrix} \cosh \lambda & 0 & \sinh \lambda \\ 0 & 1 & 0 \\ \sinh \lambda & 0 & \cosh \lambda \end{bmatrix}
\]

Check that the parameter \( \lambda \) is the hyperbolic distance of the center \( O \) to its image point \( \phi_L(O) \).
Problem 6. Given a line $l = PP'$ in the Klein disk model, and points $P$ and $P'$ on this line. Suppose that the center $O$ does not lie on this line. One can define a parallel shift along the line $l$ which maps $P$ to $P'$. Indeed, such a parallel shift is the automorphic collineation corresponding to a Lorentz boost.

Describe how to construct the image point to which the center $O$ is mapped by the parallel shift. You can use a Saccheri quadrilateral $\Box OFF'O$ with points $F$ and $F'$ on the line $l$. Provide a drawing for an adequate example.
Problem 7 (Martin’s theorem). Describe the following constructions done inside the hyperbolic plane and provide a sketch.

(a) Given is a segment $a$ and the corresponding angle of parallelism $\Pi(a) = 45^\circ$. Construct a segment $b$ with angle of parallelism $\Pi(b) = 60^\circ$.

(b) Given is a segment $a$ and the corresponding angle of parallelism $\Pi(a)$. Construct the angle of parallelism $\Pi(2a)$ of the double segment.
Figure 1: An automorphic collineation that transports points along horocycles.

**Proposition 1.** Given any ideal point $U$ on the circle of infinity and any two different points $A$ and $A'$ on the tangent to the circle $\partial D$ at point $U$. Then there exists exactly one automorphic collineation which keeps $U$ fixed, map $A$ to $A'$ and has no other fixed point. This mapping preserves the orientation of the hyperbolic plane.

**Proof of Proposition 1.** Let $X$ and $X'$ be the touching points of the tangents from points $A$ and $A'$ to $\partial D$. We let $Z'$ be the intersection point of these tangents and draw the line $b = UZ'$. Let $Y'$ be the second end of line $b$, let $Y$ be the second end of line $AY$.

Of the four points $A, X, Y, U$ nor of the four points $A', X', Y', U$ no three lie on a line. By the Main Theorem ??, there exists exactly one projective mapping taking $A \mapsto A', X \mapsto X', Y \mapsto Y'$ and leaving point $U \mapsto U$ fixed. This mapping takes the tangents to $\partial D$ at points $U$ and $X$ to the tangents at $U$ and $X'$ since $A$ is mapped to $A'$. Furthermore point $Y \in \partial D$ is mapped to $Y' \in \partial D$. Counting multiplicity, five points of the circle of infinity are mapped to five other points of this circle. Hence the prescribed mapping is an automorphic collineation.

On the other hand, the obtained mapping is unique, since any automorphic collineation with the required property necessarily maps $A \mapsto A', X \mapsto X', Y \mapsto Y'$ and leaving point $U \mapsto U$ fixed. It is left to the reader to check that $X, Y, U$ and $X', Y', U$ define the same orientation of the circle $\partial D$. \hfill \Box
Problem 8. Convince yourself that the automorphic collineation constructed above is the composition $R_b \circ R_a$ of the two reflections across lines $a$ and $b$. 
Proposition 2 (Hilbert-Klein Metric). In the Klein model, the infinitesimal hyperbolic distance $ds$ of points with coordinates $(X,Y)$ and $(X + dX, Y + dY)$ is

$$ds^2 = \frac{dX^2 + dY^2 - (XdY - YdX)^2}{(1 - X^2 - Y^2)^2}$$

Problem 9 (The Hilbert-Klein metric). Use Gauss’ characteristic equation

$$K = \frac{1}{2H} \frac{\partial}{\partial u} \left[ \frac{F}{EH} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right] + \frac{1}{2H} \frac{\partial}{\partial v} \left[ \frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{EH} \frac{\partial E}{\partial u} \right]$$

where $H = \sqrt{EG - F^2}$, and check directly that the Hilbert-Klein metric from proposition 2

$$ds^2 = \frac{dX^2 + dY^2 - (XdY - YdX)^2}{(1 - X^2 - Y^2)^2} = \frac{dr^2 + r^2(1 - r^2)d\theta^2}{(1 - r^2)^2}$$

has constant Gaussian curvature $K = -1$. We use polar coordinates $X = r \cos \theta, Y = r \sin \theta$ since this simplifies the calculation considerably.