Outline of a development of fractions

Fractions are a stumbling block for many students. In thinking about the reasons, two points present themselves as being particularly relevant:

i) To deal successfully with fractions, students must revise their conception of what a number is: from a count or additive conception to a ratio or multiplicative conception – a description of the relative size of a given quantity, as measured by another quantity, which functions as the unit. In the mathematics education literature, this is sometimes called the transition from additive thinking to multiplicative thinking.

ii) In contrast to the situation for whole numbers, we do not have unique names for rational numbers; the number indicated by any given fraction is unchanged if both the numerator and denominator are multiplied by the same factor; and doing computations with fractions often hinges on selecting an appropriate representation, which may involve changing from the representation with which one is initially presented. In particular, it is often necessary to replace two fractions with equivalent fractions that have the same denominator.

The Common Core State Standards in Mathematics attempts to ameliorate the difficulties in approaching fractions by introducing unit fractions, and emphasizing that they are new units. Concomitant to this approach is close attention to the units attached to numbers. This emphasis on units can help with both of the difficulties mentioned above. The goal of this note is to sketch a development of the domain of fractions, starting with unit fractions.

I: Meeting Fractions

A. Introduce unit fractions: \( \frac{1}{d} \) of some quantity (aka the unit, or the whole) is another quantity, such that \( d \) copies of it make the original quantity.

It is probably advisable to give many examples. There are indeed many examples available from common practice with measurement. For example, we have many units of time. We call \( \frac{1}{7} \) of a week, a “day”; we call \( \frac{1}{24} \) of a day, an “hour”. We call \( \frac{1}{60} \) of an hour a “minute”. We call \( \frac{1}{3600} \) of a minute a “second”. (And though it is not current, in Elizabethan plays, you can find \( \frac{1}{60} \) of a second - the “trice”. )

Students should realize that any convenient quantity can be considered to be the unit. Other quantities are then assigned numbers describing how many of the unit they are made of. For example, in dealing with soda, we have the can, the 6-pack and the case (4 six-packs). We could take the unit to be the can, in which case a can is \( \frac{1}{d} \), the 6-pack is \( \frac{6}{d} \) and the case is \( \frac{24}{d} \). We could take the unit to be the six-pack, in which case a can is \( \frac{1}{6} \), and a case is \( \frac{4}{6} \). We could take the unit to be the case, and then a can is \( \frac{1}{24} \), and a 6-pack is \( \frac{1}{4} \).

The variability or arbitrariness of the unit obliges us, and students in particular, to always be specific, and to state the units to which their numbers refer. Teachers should enforce this rule strictly, until satisfied that students are quite comfortable with fractions.

B: The general fraction \( \frac{n}{d} \) is defined to be \( n \) copies of the unit fraction \( \frac{1}{d} \):

\[
\frac{n}{d} = n \times \frac{1}{d}.
\]

C: Models

Many models and examples of fractions should be presented, including set models and circular (pie and pizza) models, but not only these. Especially important for thinking about fractions are the area model and the linear model or number line.

In the area model, the unit is a rectangle (aka brownie pan, or corn bread), which can be partitioned into \( d \) equal parts (subrectangles) of size \( \frac{1}{d} \) by drawing equally spaced lines parallel to a pair of opposite sides. (If the rectangle has unit area, then the area of each of the subrectangles will have area \( \frac{1}{d^2} \), but one need not invoke the idea of area, since in the main applications, one will always be comparing numbers of congruent rectangles to reach the desired conclusions.) A valuable feature of the area model is that the unit rectangle can be subdivided into pieces of size \( \frac{1}{d} \) in one direction, and pieces of size \( \frac{d}{1} \) in the other direction, and the interaction of these two operations provides valuable insight into several important situations.
The linear model takes place on an infinite half line (the number line, or number ray). There is a notion of length or distance, and in particular, there is a unit of length. Then the points on the line are labeled by the

Measurement Principle:

The number labeling a point tells how far the point is from the origin/endpoint, as a multiple of the unit distance.\(^1\)

The endpoint itself is labeled 0, since it is at 0 distance from itself. The unit interval \([0, 1]\) has unit length, and 1 is at distance 1 from 0. Then 2 is twice as far from 0 as one is, so the interval \([0, 2]\) is composed of two intervals of unit length, namely \([0, 1]\) and \([1, 2]\). And similarly with larger whole numbers.

The number \(\frac{1}{2}\) goes in the middle of the unit interval, so that \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\) have the same length, which is \(\frac{1}{2}\). The label \(\frac{1}{2}\) goes with the middle point, because then the point 1 is two times as far from 0 as \(\frac{1}{2}\) is. Similarly, the points \(\frac{1}{3}\) and \(\frac{2}{3}\) are the points that partition the interval \([0, 1]\) into three equal subintervals, so that each interval has \(\frac{1}{3}\) the length of \([0, 1]\). The label \(\frac{1}{3}\) goes with the interval that starts at 0, since the point 1 will then be 3 times as far from 0 as is \(\frac{1}{2}\). The label \(\frac{2}{3}\) goes on the other \(\frac{1}{3}\) division point, since there are two intervals of length \(\frac{1}{3}\) between it and 0, or in other words, it is 2 times as far from 0 as \(\frac{1}{3}\) is. Similar reasoning serves to locate fractions with larger denominators on the number line.

The placing of fractions on the number line is an important issue and is not straightforward for many students, so it should receive careful discussion.

The placing on the number line of the fractions \(\frac{n}{d}\) for a fixed denominator \(d\) and a numerator \(n\) that varies through all whole numbers produces a lovely and compellingly regular image, of equally spaced points partitioning the line into equal intervals of length \(\frac{1}{d}\). They look just like the whole numbers, except they are closer together, and \(d\) of the smaller intervals fit inside each unit interval.

**D: A fraction represents a division.**

Our definition of a fraction is as a multiple of a unit fraction. However it is important for students to realize that a fraction also can be interpreted as the result of a division:

\[
\frac{n}{d} = n \div d.
\]

This should be illustrated with several examples. Both linear and area models can be used effectively to show why this is true.

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\(^{1}\) Getting students comfortable with this principle takes substantial attention to the ideas of measurement. See *Three Pillars of First Grade Mathematics* (commoncoretools.me/wp-content/uploads/2012/02/3pillars.pdf) for suggestions about how to start.
II: Arithmetic of fractions with fixed denominator

Once students are reasonably familiar with the idea of a fraction, they can begin to study their arithmetic. The first stage is to deal with the case of fractions with the same denominator.

A: Addition

The general formula is
\[ \frac{n}{d} + \frac{m}{d} = \frac{n + m}{d}. \]

Examples of this rule can be justified directly from the definition, and illustrated with various models, including linear and area models. An attractive feature of the linear model is that, just as addition of whole numbers can be interpreted in terms of length by placing bars end-to-end, this works equally well for fractions. Thus, although the symbolic calculation of the sum of fractions may seem quite different from that of whole numbers, the associated geometry is the same. This remains true for the case of adding fractions with unequal denominators as well.

B: Comparison

The general rule is
\[ \frac{n}{d} \geq \frac{m}{d} \quad \text{if and only if} \quad n \geq m. \]

This again can be illustrated by example in various models, and has a natural interpretation in terms of length.

C: Multiplication by a whole number

The general formula is
\[ m \times \frac{n}{d} = \frac{mn}{d}. \]

Similarly to addition and comparison, this can be illustrated by example with various models. It is consistent with the conception of multiplication by a whole number as repeated addition.
III: Relating fractions with different denominators

The key to working successfully with fractions is to be able to accommodate fractions with different denominators. This is based on several key relationships, which can be convincingly illustrated with the area model, and also, sometimes with perhaps more effort, with the linear model. These ideas may present difficulties for some students, but since they are critical for doing and understanding calculations, they should be presented carefully and discussed, with many examples, to promote acceptance.

A: Repeated Subdivision

As a quantity in its own right, \(\frac{1}{d}\) can itself be divided into fractions of itself. Thus we can consider \(\frac{1}{e}(\frac{1}{d})\). This is not a new entity, but in fact is a unit fraction of the original quantity. Precisely,

\[
\frac{1}{e}(\frac{1}{d}) = \frac{1}{ed}.
\]

This can be illustrated with the area model by dividing the unit rectangle into \(d\) equal vertical strips, and \(e\) equal horizontal strips. Then the lines defining the horizontal strips also divide each vertical strip into \(e\) equal pieces, which are therefore \(\frac{1}{e}(\frac{1}{d})\). But also, all the small pieces of all the strips are congruent, and there are \(ed\) of them in the whole rectangle, so each one constitutes \(\frac{1}{ed}\) of the whole.

B: Reconstitution

In the same picture as used to demonstrate repeated subdivision, we see that the vertical strips of size \(\frac{1}{d}\) are composed of \(e\) of the small rectangles of size \(\frac{1}{ed}\); and likewise, each of the horizontal strips of size \(\frac{1}{e}\) is composed of \(d\) of the small rectangles. This illustrates the relationships

\[
\frac{1}{d} = e \times \frac{1}{ed} = \frac{e}{ed}; \quad \text{and} \quad \frac{1}{e} = d \times \frac{1}{ed} = \frac{d}{ed}.
\]

C: Renaming

If we multiply the reconstitution identity by a whole number \(n\), we obtain

\[
\frac{n}{d} = n \times \frac{1}{d} = n \times \frac{e}{ed} = \frac{ne}{ed}.
\]

This shows that the number represented by the fraction \(\frac{n}{d}\) can also be represented (in infinitely many ways) by other fractions, obtained by multiplying both the numerator and denominator by the same number. This means, that in any situation where \(\frac{n}{d}\) appears in a calculation, it can be replaced by \(\frac{ne}{de}\). This process is sometimes called renaming the fraction. Also, the fraction \(\frac{ne}{de}\) is said to be equivalent to \(\frac{n}{d}\).

The renaming process can be illustrated and justified with the area model by taking the same picture as used to argue for repeated subdivision, and identifying the area representing \(\frac{n}{d}\) (which might fill up several unit rectangles if \(n > d\)) as a union of the rectangles of size \(\frac{1}{de}\).

D: Renaming to find a Common Denominator

In particular, the renaming process shows that, given any two fractions \(\frac{n}{d}\) and \(\frac{m}{e}\), we can rename them both as fractions with the same denominator:

\[
\frac{n}{d} = \frac{ne}{de}, \quad \text{and} \quad \frac{m}{e} = \frac{md}{ed} = \frac{md}{de}.
\]

This is essential for doing arithmetic with fractions.
IV: Arithmetic of general fractions

A: Addition

The general formula for the sum of two fractions is reduced to the already known formula for addition of fractions with the same denominator by renaming the fractions to have a common denominator:

\[
\frac{n}{d} + \frac{m}{e} = \frac{ne}{de} + \frac{md}{de} = \frac{ne + md}{de}.
\]

This can be described verbally: multiply the numerator and denominator of each fraction by the denominator of the other one, then add the numerators of the resulting fractions, and put the sum over the product of the two denominators.

This can be illustrated by producing the area model for each renamed fraction, and noting that the sum of the two is just represented by the union of the two regions representing the two fractions. Each fraction should be created within its own unit rectangle (or union of several, if it is larger than 1). The rectangles containing each fraction can be lined up adjacent to each other.

This can also be illustrated with the number line. A particularly satisfactory feature of adding fractions on the number line is that the principle that governed whole number addition - that you add by juxtaposing intervals with lengths equal to the addends, (with one length having its left endpoint at the origin), is equally valid for fractions.

B: Comparison

Given two fractions, \( \frac{n}{d} \) and \( \frac{m}{e} \), we can compare them by renaming them as fractions with the same denominator, and comparing the numerators:

\[
\frac{n}{d} = \frac{ne}{de} \geq \frac{md}{de} = \frac{m}{e}
\]

if and only if \( ne \geq md \).

This is often called the cross multiplication criterion: multiply the numerator of each fraction by the denominator of the other, and compare the products. Although it is justified by the renaming process, it can also be demonstrated using the area model, subdivided as if preparing to rename.

C: Multiplication

Multiplication can also be represented nicely using the picture for repeated subdivision, If the unit rectangle is subdivided vertically into strips of size \( \frac{1}{d} \) and horizontally into strips of size \( \frac{1}{e} \), then one sees that that the rectangle formed by \( n \) vertical strips is also subdivided into equal horizontal strips by the same lines as subdivide the whole rectangle. Thus, the intersection of on of these strips with the rectangle representing \( \frac{n}{d} \) amounts to \( \frac{1}{e} \) of \( \frac{n}{d} \), so \( m \) of these strips constitute \( \frac{m}{e} \times \frac{n}{d} \). But this is just the intersection of the rectangles representing \( \frac{n}{d} \) and \( \frac{m}{e} \), and it consists of \( nm \) of the small rectangles of size \( \frac{1}{de} \), so amounts to \( \frac{nm}{de} \) of the whole. This gives the multiplication formula:

\[
\frac{n}{d} \times \frac{m}{e} = \frac{nm}{de}.
\]

Of course, this can be described verbally by saying: the numerator of the product is the product of the numerators, and the denominator of the product is the product of the denominators; or more telegraphically, multiply the numerators, and multiply the denominators. Although this reasoning works in general, it is easier to execute when the fractions are less than 1 (\( n < d \) and \( m < e \)).

D: Division

Division is the most difficult of the operations to understand for whole numbers, In fact, it is not actually an operation in the strict sense, since not all divisions of whole number produce whole number results. Fractions were invented to allow division of any two whole numbers. One of the great benefits of extending the number system from whole numbers to fractions is that division becomes a true operation: fractions not only give the answers for dividing one whole number by another, but the result of dividing any
fraction by any fraction is also another fraction. It is not necessary to invent even more numbers to allow for division of a fraction by a fraction.

More remarkably, the operations of division and multiplication are merged into a single operation: division by the fraction $\frac{e}{m}$ is the same as multiplying by another fraction, usually called the **reciprocal** of $\frac{e}{m}$. Moreover there is a simple formula for the reciprocal: it is $\frac{m}{e}$. Thus, the formula for division of fraction takes the simple form:

$$\frac{n}{d} \div \frac{e}{m} = \frac{n}{d} \times \frac{m}{e} = \frac{ne}{dm}.$$  

This rule is usually described as “invert and multiply”.

Because of the difficulty associated with understanding division, we will give several arguments for the invert-and-multiply rule.

i) **Measurement.** The measurement interpretation of whole number division $n \div d$, when it is possible, is, how many of quantity $d$ fit in quantity $n$? We can ask the same question of two fractions: how many copies of $\frac{m}{e}$ fit in $\frac{n}{d}$? To answer this question, we should express both $\frac{n}{d}$ and $\frac{m}{e}$ in the same units, which means putting them over a common denominator. Thus, we should change our question to: how many copies of $\frac{dm}{de}$ fit in $\frac{ne}{de}$? But we can now recognize this question as being a whole number division question, but with the units being the unit fraction $\frac{1}{de}$. That is, we are asking, how many groups of size $\frac{md}{de}$, of units of size $\frac{1}{de}$, fit into a group of size $\frac{ne}{de}$ of the same units? And we know that the answer to this question is just the fraction $\frac{ne}{dm}$. The thinking involved in this argument can be well illustrated with the area model.

ii) **Missing factor.** In whole number situations, the quotient $\frac{a}{b}$ can also be understood as a missing factor: it is the number $x$ which, if multiplied by $b$, gives $a$. In other words, $\frac{a}{b}$ is the solution $x$ to the equation

$$bx = a.$$  

If we take this as defining the quotient also when $a$ and $b$ are fractions, we are looking for $x$ such that

$$\left(\frac{m}{e}\right)x = \frac{n}{d}.$$  

If we use the formula for multiplication of fractions, we can check that $\frac{ne}{dm}$ satisfies this equation. Indeed,

$$\frac{m}{e} \times \frac{ne}{dm} = \frac{men}{edm} = \frac{n(me)}{d(me)} = \frac{n}{d},$$

as desired. In this calculation, we used the commutative and associative rules for multiplication (for ordinary whole numbers) to rearrange the products in the numerator and the denominator, and then we used the renaming process to simplify the fraction.

iii) **Undoing multiplication.** Sometimes division is presented as the undoing of multiplication. If we multiply the whole number $m$ by the whole number $n$, and then divide the product by $m$, we just get back to $n$. We can check by multiplication that the product of $\frac{ne}{dm}$ and $\frac{m}{e}$ is 1:

$$\frac{m}{e} \times \frac{e}{m} = \frac{me}{em} = \frac{me}{me} = 1.$$  

It follows from this that multiplication by $\frac{m}{e}$ does indeed undo multiplication by $\frac{e}{m}$:

$$\frac{e}{m} \times \left(\frac{m}{e} \times \frac{n}{d}\right) = \frac{e}{m} \times \frac{mn}{ed} = \frac{e(mn)}{m(ed)} = \frac{n(me)}{d(me)} = \frac{n}{d}.$$  

In this calculation, we have again used the formula for multiplication, plus the commutative and associative rules for multiplication (of whole numbers).

This calculation shows that multiplication by $\frac{e}{m}$ does undo multiplication by $\frac{e}{m}$, and so should be considered as division by $\frac{e}{m}$. Thus, division by a fraction is the same as multiplication by another fraction. The fraction $\frac{e}{m}$ is known as the **reciprocal** of $\frac{m}{e}$. It is obtained by exchanging the numerator and denominator of $\frac{m}{e}$. This is what gives rise to the familiar directive: to divide by $\frac{m}{e}$, invert and multiply.
V: Refinements

After students are comfortable with the basic aspects of fractions, as sketched above, including their arithmetic, several topics that are less central but which promote a more refined view of fractions should probably be covered. The main ones are mixed numbers, and “reducing fractions”, which actually is about finding the simplest form of a rational number - finding the smallest numerator and/or denominator that can be used to represent the number. It is not at all obvious that there is a unique simplest form for a given rational number, but there is, and its existence is related to some basic number theory, especially the greatest common divisor (GCD), the Euclidean Algorithm and uniqueness of prime factorization (which was dubbed the Fundamental Theorem of Arithmetic by Gauss). The Euclidean Algorithm is a general method to find the greatest common divisor of two whole numbers; it was described by was given by Euclid in his Elements. We will discuss these theoretical issues briefly below.

A: Mixed numbers and improper fractions

The discussion above has more or less ignored the fact that fractions are usually thought of as being less than one. For a given denominator \(d\), the numerator \(n\) was freely allowed from the beginning to range through all whole numbers; and indeed, this was a key feature of the number line representation of the system of all fractions \(\frac{n}{d}\) for fixed \(d\), as defining a regular division of the number line into intervals of size \(\frac{1}{d}\)

However, it is not straightforward to tell the size of a fraction. When the numerator \(n\) increases and the denominator \(d\) is fixed, the fraction increases, but when \(n\) is fixed and \(d\) increases, the fraction decreases. These opposite tendencies can make it not so simple to determine the size of fractions. For that reason, among others, it can be useful to express a fraction as the sum of a whole number and a fraction less than one. If the whole number is non-zero, it will constitute the main part of the fraction, and the part less than one will be a relatively small correction. A fraction between 0 and 1 is called a proper fraction, and the expression of a fraction as a sum \(\frac{n}{d} = q + \frac{r}{d}\) of a whole number and proper fraction is called a mixed number. The traditional terminology for a fraction \(\frac{n}{d}\) with \(n > d\) is improper fraction, but we do not advocate for this usage.

To obtain the expression of \(\frac{n}{d}\) as a mixed number, one performs the division-with-remainder algorithm of \(n\) by \(d\), and find the DWR quotient \(q\) of \(n\) by \(d\), and the remainder \(r\): \(n = qd + r\), with \(r < d\) \(\iff \frac{n}{d} = q + \frac{r}{d}\).

The arithmetic of mixed numbers is somewhat involved, which is not surprising, since two mixed numbers involves dealing with three different units: the whole, and the unit fractions corresponding to the two denominators. However, there is nothing conceptually new involved, and the computations can be carried out on the basis of the formulas for arithmetic of part III above, Accordingly, we will not describe the details.

B: Simplest form, GCD

The fact that fractions have many names gives rise to a problem: given a fraction \(\frac{n}{d}\), can we find another fraction \(\frac{n'}{d'}\) that equals \(\frac{n}{d}\), but is simpler in the sense that \(n' < n\) and \(d' < d\)? If we can, we generally use \(\frac{n'}{d'}\) in place of \(\frac{n}{d}\), especially for purposes of exposition, since we tend to find fractions with smaller denominators easier to interpret. Thus, we would find it easier to process a statement that \(\frac{1}{6}\) of students plan to study more mathematics, than that \(\frac{37}{222}\) students plan to study more mathematics.

The problem of simplifying fractions amounts to finding a common factor or common divisor of the numerator and denominator; for if \(f > 1\) divides both \(n\) and \(d\), that is, \(n = nf\) and \(d = df\), then \(\frac{n}{d} = \frac{n'}{d'}\) that is, \(\frac{n}{d}\) can be simplified.
It is a remarkable and convenient fact that every pair \{n_1, n_2\} of whole numbers has a \textit{greatest common divisor} (GCD, or GCF). This is a common divisor \(f_o\), which is greatest not only in the sense that it is larger than any other common divisor \(f\), but is also largest multiplicatively, in the sense that, if \(f\) divides both \(n_1\) and \(n_2\) exactly, it also divides \(f_o\) exactly: \(f_o = f f'\), for some whole number \(f'\).

The existence of a greatest common divisor guarantees a “best possible” simplification of any fraction. Given a fraction \(\frac{a}{d}\), if \(f_o\) is the GCD of \(n\) and \(d\), with \(n = nf_o\) and \(d = df_o\), then \(\frac{a}{d} = \frac{n}{d}f_o\), that is, represents the same number. Moreover, any other fraction that represents the same number will have the form \(\frac{a}{d} = \frac{n}{df_o}f\), for some whole number \(f\) (which is then seen to be the GCD of \(n\) and \(d\). Thus, every fraction has a unique “simplest form”, or “reduced form”, in which the numerator and denominator have no common factors greater than 1. We then say that the numerator and denominator are \textit{relatively prime}, and that the fraction is \textit{reduced}.

These facts may or may not be clearly stated in K-12 math instruction. Even if they are clearly stated, it is unlikely that they are proved. In particular, it is rarely if ever done in K-12 mathematics to show the existence of the GCD, outside of a few examples. The usual approach to finding GCDs is by means of prime factorization. However, beyond a rather limited range, prime factorization is difficult to do: no efficient algorithm is known. There is, however, an efficient way to find the GCD of two numbers. It was described by Euclid in Book VII of his \textit{Elements}. The Euclidean Algorithm is pretty much ignored in K-12 mathematics. (Perhaps it is presented in specialized courses at some elite high schools.) We discuss it briefly below.

C: LCM

When adding fractions, we need to express them both in the same units, which means renaming them as fractions with a the same denominator. This is traditionally called a “common denominator”. Common denominators figured also in our discussion of comparison and division. In discussing renaming in part II, we showed that, given any two fractions, the product of their denominators could always be used as a common denominator. However, there are situations where smaller numbers can also be used. For example \(\frac{3}{4} = \frac{9}{12}\) and \(\frac{5}{6} = \frac{10}{12}\). The question thus arises, given two fractions, what is the smallest number that can serve as a common denominator for fractions equivalent to both of them?

If we assume that \(\frac{a}{d}\) is in lowest terms, then by part A, we know that any fraction equivalent to \(\frac{a}{d}\) must have denominator divisible by \(d\). Similarly, if \(\frac{m}{e}\) is another fraction in lowest terms, any renaming of it must have denominator divisible by \(e\). On the other hand, if \(r = ad = be\) (for whole numbers \(a\) and \(b\)) is a multiple of both \(d\) and \(e\), then \(\frac{a}{d} = \frac{an}{ad} = \frac{an}{r}\) and \(\frac{m}{e} = \frac{bm}{be} = \frac{bm}{r}\), so \(r\) can serve as a common denominator for both fractions.

We call the number \(r = ad = be\) a \textit{common multiple} of \(d\) and \(e\). The product \(de\) clearly is a common multiple of \(a\) and \(b\), but it may happen that smaller numbers are also common multiples. For example, 12 is a common multiple of 4 and 6, which allowed the renaming above of \(\frac{3}{4}\) and \(\frac{5}{6}\) in terms of \(\frac{1}{12}\)s.

The smallest positive multiple of two whole numbers is called their \textit{least common multiple} (LCM). It can be shown that \(LCM(d, e)\), the least common multiple of \(d\) and \(e\) is unique, and is multiplicatively minimal, in the sense that it divides any other common multiple. Moreover, the LCM is related to the GCD by an elegant formula:

\[
LCM(d, e) \cdot GCD(d, e) = de.
\]

Sometimes, when instruction puts heavy emphasis on putting fractions in simplest terms, students may be told that when adding fractions, they should find the LCM of the denominators. While this may somewhat relieve the computational burden in dealing with the sum, finding the LCM of two numbers is not a simple process. Indeed, it typically is considerably more work than finding the sum. Moreover, the conceptual issues involved in finding the LCM are much more challenging than the relatively straightforward formula for the sum of two fractions, as given in part III. Thus, it is not recommended to raise the issue of LCM while students are learning the basics of fractions. It can be discussed later, as a refinement, perhaps for students with strong interest in mathematics.
VI. Theoretical Addendum: Euclidean Algorithm and Prime Factorization

A: The Euclidean Algorithm

Theoretical justification for the existence of the GCD and uniqueness of prime factorization is provided by the Euclidean Algorithm (EA). Although it is avoided in school mathematics, we will give a brief discussion of it here.

The EA is an extension form of the procedure for division-with-remainder, which is often called the division algorithm (DA). The DA expresses a given number \( n \) as a multiple of a given number \( d \), plus a remainder \( r \), which is required to be less than \( d \):

\[
 n = qd + r, \quad \text{with} \quad r < d. \tag{DA}
\]

The condition on \( r \) is equivalent to saying that \( q \) is the largest number such that \( qd \) is less than or equal to \( n \). We divide \( n \) into as many pieces of size \( d \) as we can, and then \( r \) is what is left, too little for form another piece of size \( d \). This conception of division depends strongly on the ordering of the whole numbers.

The EA starts with two numbers \( n_1 \) and \( n_2 \), with \( n_2 \) taken to be the smaller. It performs the DA of \( n_1 \) by \( n_2 \). The remainder is called \( n_3 \). Then the DA is performed for \( n_2 \), and \( n_3 \) (which is guaranteed to be less than \( n_2 \) by the DA). This process is continued until a remainder of 0 is obtained; and then it stops. Since the remainders form a strictly decreasing sequence, this must happen at some step. Thus, if the EA stopped at the fourth step, it would look like this:

\[
egin{align*}
 n_1 &= q_1n_2 + n_3, \\
 n_2 &= q_2n_3 + n_4, \\
 n_3 &= q_3n_4 + n_5, \\
 n_4 &= q_4n_5.
\end{align*} \tag{EA}
\]

The main statement about the EA is this:

**Theorem:** The last non-zero remainder in the EA is the GCD of \( n_1 \) and \( n_2 \).

We will explain why this is true.

The theorem makes two claims:

i) The last non-zero remainder divides the two starting numbers.

ii) Any number that divides both of the two starting also divides the last remainder.

These are the defining properties of GCD. Thus, the last non-zero remainder is the GCD of the two starting numbers; this is what the EA does.

To show ii), one works forward using the successive equations of the EA. We need the idea of an integral combination of two numbers \( r \) and \( s \). This is a sum

\[ ar + bs, \]

where \( a \) and \( b \) are integers (whole numbers, perhaps with a - sign). We take as obvious the following facts:

a) An integral combination of whole numbers is an integer.

b) An integral combination of two integral combinations of two numbers is again an integral combination of the original two numbers.

c) If \( r \), \( s \) and \( t \) are whole numbers, and \( t \) divides \( r \) exactly and divides \( s \) exactly, then \( t \) also divides exactly any integral combination of \( r \) and \( s \).

Now return to study the EA. The \( k \)-th equation of the EA has the form

\[ x_k = q_kx_{k+1} + x_{k+2}. \]

We can rewrite this to express the \( n_{k+2} \) in terms of \( n_{k+1} \) and \( n_k \):

\[ x_{k+2} = x_k - q_kx_{k+1}. \]
This equation shows that \( x_{k+2} \) is an integral combination of \( x_{k+1} \) and \( x_k \). If we work forward from \( x_1 \) and \( x_2 \), using fact b) above, we conclude first that \( x_3 \) is an integral combination of \( x_1 \) and \( x_2 \), and second that \( x_4 \) is an integral combination of \( x_1 \) and \( x_2 \), and third that \( x_5 \) is an integral combination of \( x_1 \) and \( x_2 \). Continuing like this, we conclude that all the \( x_k \) are integral combinations of \( x_1 \) and \( x_2 \). In particular, the last non-zero remainder is an integral combination of \( x_1 \) and \( x_2 \). It therefore follows from statement c) above that all the remainders are divisible by any number that divides \( x_1 \) and \( x_2 \). In particular, the last non-zero remainder is divisible by \( x_1 \) and \( x_2 \). This concludes the proof of claim ii) above.

To show i) one works backward through the same set of equations. Observe that the \( k \)-th equation of the EA expresses \( x_k \) as an integral combination of \( x_{k+1} \) and \( x_{k+2} \). If the last non-zero remainder is \( x_ℓ \), then the \( ℓ-1 \)-th equation says \( x_{ℓ−1} = q_{ℓ−1}x_ℓ \), which means that \( x_{ℓ−1} \) is divisible by \( x_ℓ \). Since the \( ℓ−2 \)-th equation expresses \( x_{ℓ−2} \) as an integral combination of \( x_{ℓ−1} \) and \( x_ℓ \), we conclude that \( x_{ℓ−2} \) is also divisible by \( x_ℓ \) by fact c) above. Then since \( x_{ℓ−3} \) is an integral combination of \( x_{ℓ−2} \) and \( x_{ℓ−1} \), we can again use fact c) to conclude that \( x_{ℓ−3} \) is divisible by \( x_ℓ \). Working backward like this, we eventually conclude that \( x_1 \) and \( x_2 \) are divisible by \( x_ℓ \). This concludes the proof of claim i) above. So both claims are true.

Here is the example of working backward in the 4-step algorithm stated above;

\[
\begin{align*}
n_4 &= q_1n_5, \\
n_3 &= q_3n_4 + n_5 = q_3(q_4n_5) + n_5 = (q_3q_4 + 1)n_5, \\
n_2 &= q_2n_3 + n_4 = q_2((q_3q_4 + 1)n_5) + q_4n_5 = (q_2q_3q_4 + q_2 + q_4)n_5, \\
n_1 &= q_1n_2 + n_3 = q_1(q_2q_3q_4 + q_2 + q_4)n_5 + (q_3q_4 + 1)n_5 \\
&= (q_1q_2q_3q_4 + q_1q_2 + q_1q_4) + (q_3q_4 + 1))n_5 = (q_1q_2q_3q_4 + q_1q_2 + q_1q_4 + q_3q_4 + 1)n_5.
\end{align*}
\]

And here is the example of working forward:

\[
\begin{align*}
n_3 &= n_1 - q_1n_2, \\
n_4 &= n_2 - q_2n_3 = n_2 - (n_1 - q_1n_2) = (q_1 + 1)n_2 - n_1, \\
n_5 &= n_3 - q_3n_4 = (n_1 - q_1n_2) - q_3((q_1 + 1)n_2 - n_1) = (q_3 + 1)n_1 - (q_1q_3 + q_1 + q_3)n_2.
\end{align*}
\]

B: Uniqueness of Prime Factorization

Recall that a prime number is a whole number that is not divisible by any number strictly less than itself and greater than 1. The first so many primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, . . . . Since two is the first prime, all primes greater than 2 are odd numbers.

The prime numbers are the generators for the whole numbers under multiplication, as described by the following two facts.

i) (Existence of prime factorization) Every whole number can be expressed as a product of prime numbers.

This is easy to see, using the following principle, which is a basic property of the whole numbers, and is the basis for mathematical induction:

Every non-empty set of whole numbers has a smallest element.

Accepting this principle, suppose there are numbers that are not expressible as products of prime numbers, and let \( n_o \) be the smallest one. Then either \( n_o \) is prime, or it isn’t. If it is prime, it is its own expression of itself as a product of primes. If it is not prime, it is expressible as a product

\[
n_o = ab,
\]

where \( a \) (and hence, also \( b \)) is a number strictly between 1 and \( n_o \). So both \( a \) and \( b \), being smaller than \( n_o \), are expressible as products of primes. Then multiplying these two expressions together gives an expression for \( n_o \) as a product of primes, which is a contradiction. Hence, the set of numbers not expressible as a product of primes must be empty; in other words, all numbers are expressible as a product of primes, as claimed.

ii) (Uniqueness of prime factorization) A whole number has only one factorization into a product of primes, up to reordering of the primes. Thus

\[
n = 2^{m_2(n)} \cdot 3^{m_3(n)} \cdot 5^{m_5(n)} \cdot \ldots,
\]
where all the exponents $m_p(n)$ are uniquely determined by $n$.

The proof again is an argument by contradiction. Suppose there are numbers with more than one prime factorization, and let $n_o$ be the smallest one. Then $n_o$ must not be prime, or it would be its own unique prime factorization. Suppose

$$ n_o = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_j $$

is one factorization of $n_o$ into primes, and

$$ n_o = p'_1 \cdot p'_2 \cdot p'_3 \cdot \ldots \cdot p'_k $$

is another one. Then all the primes in the two factorizations must be different. For if there were one prime factor in common, say $p_1 = p'_1$, after reordering for sake of convenience, then

$$ \frac{n_o}{p_1} = p_2 \cdot p_3 \cdot \ldots \cdot p_j = p'_2 \cdot p'_3 \cdot \ldots \cdot p'_k $$

would be two prime factorizations of $\frac{n_o}{p_1}$. But $\frac{n_o}{p_1}$, being smaller than $n_o$, can have only one prime factorization, so the $p_c$ must be the same as the $p'_d$ up to reordering; which means that the two factorizations of $n_o$ were also the same, up to reordering.

So if we have the two prime factorizations for $n_o$, the prime $p_1$ must not be equal to any of the $p'_b$. Now write

$$ n_o = p'_1 \cdot m, $$

where $m$ is the product of all the rest of the primes in the second factorization of $n_o$. Then $p_1$ does not divide either of $p'_1$, and it must not divide $m$ either, since $m < n_o$ has a unique prime factorization, which does not include $p_1$. However, this contradicts the following

Lemma: If a prime $p$ divides a product $a \cdot b$, then it divides $a$, or it divides $b$.

Proof: Suppose that $p$ does not divide $a$. Then the GCD of $p$ and $a$, since it must divide $p$, and cannot be $p$ (since $p$ does not divide $a$, by assumption), can only be 1. By the EA, this means that 1 is an integral combination of $p$ and $a$; so we have an equation

$$ rp + sa = 1, $$

for some integers $r$ and $s$.

Multiply this equation by $b$:

$$ brp + sab = b. $$

By assumption, $p$ divides $ab$: $ab = cp$ for a whole number $c$. Therefore

$$ b = brp + sab = brp + scp = (br + sc)p. $$

This says that $p$ does divide $b$, so the lemma is proved.

Since the conclusion we arrived at by assuming the existence of numbers with more than one prime factorization contradicted the lemma, it must be false. So all numbers have unique factorizations.