1 Introduction

In recent years, mathematics educators have begun to realize that understanding fractions and fractional arithmetic is the gateway to advanced high school mathematics. Yet, US students continue to do poorly when ranked internationally on fraction arithmetic. For example, consider this problem the 1995 TIMSS Trends in International Mathematics and Science Study:

What is 3/4 + 8/3 + 11/8? The options were
A. 22/15 B. 43/24 C. 91/24 D. 115/24
More than 42 percent of US eighth graders who worked this problem chose option A. This percent was exceeded only by England. Singapore, Japan, and Belgium all had fewer than 10 percent students with answer A.

Consider the following real situation. A man bought a TV set for $1000, only to return a week later, complaining to the manager that he had bought the sets only because of the advertisement that the price was set ‘at a fraction of the Manufacturers’ Suggested Retail Price (MSRP)’. But the man had seen on the internet that the MSRP was only $900. Well, the manager said, ‘ten ninths is a fraction.’ The story does not end there. The customer won a small claims decision because the judge ruled that the manager was a crook. The common understanding of fraction, the judge said, is a number less than 1.

In the words of comedian Red Skelton: Fractions speak louder than words.

What is important here is to note that CCSS requires a model for fraction that enables both conceptual understanding and computational facility. The suggested model is to first understand unit fractions in more or less the same way we understand place value numbers in building an understanding of decimal representation. A place value number (or special number), is simply a digit times a power of 10. These place value numbers are the atoms of the number system, and we learn how to build the other numbers from them and how these numbers enable computation. Now for unit fractions, we can say that every fraction (at the elementary stage we still do not make the distinction between fraction and rational number as we do below) is a sum of unit fractions:

\[
\frac{m}{n} = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}.
\]

This is similar to the idea that every positive integer is expressible as a sum of place value numbers. Once we learn to perform the arithmetic operations of place value numbers, we are led naturally to the general case. Such happens with fractions.

\[^{1}\text{Unauthorized reproduction/photocopying prohibited by law}^\text{©}\]
also. For a discussion of place value see the paper I co-authored with Roger Howe: http://www.teachersofindia.org/en/article/five-stages-place-value.

2 Key Words

Fraction, mediant, rational number, unit fraction, continued fraction, floor function, ceiling function.

3 Fraction versus rational number.

What’s the difference? It’s not an easy question. In fact, the difference is somewhat like the difference between a set of words on one hand and a sentence on the other. A symbol is a fraction if it is written a certain way, but a symbol that represents a rational number is a rational number no matter how it is written. Here are some examples. The symbol $\frac{1}{\pi}$ is a fraction that is not a rational number. On the other hand $\frac{2}{3}$ is both a fraction and a rational number. Now 0.75 is a rational number that is not a fraction, so we have examples of each that is not the other. To get a little deeper, a fraction is a string of symbols that includes a fraction bar, a numerator and a denominator. These items may be algebraic expressions or literal numbers. Any real number can be written as a fraction (just divide by 1). But whether a number if rational depends on its value, not on the way it is written. What we’re saying is that in the case of fractions, we are dealing with a syntactic issue, and in case of rational numbers, a semantic issue, to borrow two terms from computer science. For completeness, we say that a number is rational if it CAN be represented as a quotient of two integers. So 0.75 is rational because we can find a pair of integers, 3 and 4, whose quotient is 0.75.

Here’s another way to think about the difference. Consider the question ‘Are these numbers getting bigger or smaller?’

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}. \]

This apparently amusing question can provoke some serious questions about what we mean by the word ‘number’. Indeed, there are two aspects of numbers that often get blurred together: the value of a number and the numeral we write for the number. By ‘number’ we usually mean the value while the word ‘numeral’ refers to the symbol we use to communicate the number. So the numbers above are getting smaller while the numerals are getting bigger. This contrast between symbol and substance also explains the difference between rational number and fraction. A fraction is a numeral while a rational number is a number. Technically, a rational number is a collection of equivalent fractions, equivalent in the sense that $a/b \equiv c/d$ if $ad = bc$. 

\[ 1 \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}. \]
For example the rational number 0.5 is the set \( \{ a/2a \mid a \text{ is any non-zero integer} \} \). Having said all this about fraction versus rational number, in schools these often mean the very same thing. This actually makes sense because precollege students rarely discuss irrational numbers.

This essay is intended to help mathematicians interested in working with teachers and ambitious teachers working independently who want to develop a deeper understanding of some fundamental ideas. We do this by posing and solving the following problem: If \( e/f \) is a positive rational number reduced to lowest terms, we call \( e + f \) the size of \( e/f \). If \( 0 \leq a/b < c/d \) are given rational numbers (reduced to lowest terms), what is the smallest size fraction \( e/f \) between \( a/b \) and \( c/d \)? The question was brought to my attention by high school math teacher Mak Fung from Hong Kong. We give two methods including a continued fraction method for finding \( e/f \). At the end of the paper, we use the ideas developed here to generalize this problem in two different ways. Also, at the end, we also pose several more problems. These problems are not intended to be used in a classroom with young students. On the other hand, they can all be used to deepen teachers’ understanding of fractions and fractional arithmetic.

### 4 The Main Problem

The Common Core State Standards proposes a model for fractions that enables both conceptual understanding and computational facility. The suggested model begins with unit fractions. A unit fraction has numerator 1 and denominator a positive integer. Unit fractions play in the area of fractions more or less the same role that place value numbers play in building decimal representations and computation. See [?]. A fraction is then defined as a sum of unit fractions. See [?] and [?]. There are two important types of problems related to fractions. One is fractional arithmetic and the other is comparison of fractions. Both these types are addressed here. In this paper, we give two solutions (including a continued fraction solution) to the following equivalent Problems 1,1’. In Section 4, we first prove that the Problems 1,1’ are equivalent problems. In Section 4, we also give an elementary solution to Problem 1. We also prove Theorem 1 which states that the solutions to problems 1, and1’ are the same. Theorem 1 is the foundation of this paper. At the end of the paper, we generalize Problems 1, 1’ and Theorem 1 in two different ways by using the ideas given in Section 4.

**Problem 1.** Suppose \( 0 \leq \frac{a}{b} < \frac{c}{d} \) where \( \frac{a}{b}, \frac{c}{d} \) are fractions that are reduced to lowest terms and \( 0 = \frac{0}{1} \). Find a fraction \( \frac{e}{f} \) (reduced to lowest terms) such that \( 0 \leq \frac{e}{f} < \frac{c}{d} \) and such that \( f \) is a positive integer that has the smallest possible value. For this smallest possible positive integer \( f \), let us compute the smallest
possible positive integer \( e \) that can go with \( f \) so that \( 0 \leq \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \).

**Problem 1’** Suppose \( 0 \leq \frac{a}{b} < \frac{c}{d} \) where \( \frac{a}{b}, \frac{c}{d} \) are fractions that are reduced to lowest terms and \( 0 = \frac{a}{b} = \frac{0}{1} \). Find a fraction \( \frac{e}{f} \) (reduced to lowest terms) such that \( 0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d} \) and such that \( \overline{e} \) is a positive integer that has the smallest possible value. For this smallest possible positive integer \( \overline{e} \), let us compute the smallest possible positive integer \( \overline{f} \) that can go with \( \overline{e} \) so that \( 0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d} \).

**Theorem 1.** Suppose \( \frac{a}{b} = \frac{c}{d} \) have the same values in both Problems 1, 1’. Then the solutions \( (e, f, \frac{e}{f}) \) and \( (\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}) \) in Problems 1, 1’ are the same solution. That is \( (e, f, \frac{e}{f}) = (\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}) \).

## 5 An Elementary Solution to Problems 1, 1’ and a Proof of Theorem 1

We first show that Problems 1, 1’ are equivalent problems. First, we note that the solutions to \( 0 = \frac{a}{b} < \frac{c}{d} \) and \( 0 = \frac{a}{b} < \frac{e}{f} < \frac{c}{d} \) are \( e = \overline{e} = 1 \) and \( f = \overline{f} \) where \( f = \overline{f} \) is the smallest positive integer that satisfies \( \frac{a}{b} < f = \overline{f} \). Next, we note that \( 0 < \frac{a}{b} < \frac{e}{f} < \frac{c}{d} \) is true if and only if \( 0 < \frac{d}{c} < \frac{f}{e} < \frac{b}{a} \) which is the same form as \( 0 < \frac{a}{b} < \frac{e}{f} < \frac{c}{d} \). Note that \( e \) and \( f \) are essentially interchanged. Therefore, Problems 1, 1’ are equivalent problems.

Solution to Problem 1. We note that \( 0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d} \) is true if and only if \( 0 \leq \frac{ae/b}{d} < e < \frac{cf/d}{d} \) where \( e \) is an integer lying strictly between \( \frac{ae/b}{d} \) and \( \frac{cf/d}{d} \). Note that if \( 0 \leq \frac{ae/b}{d} < e < \frac{cf/d}{d} \) then \( 0 \leq \frac{a}{b} < \frac{e}{f} < \frac{c}{d} \). Also, the length of the integral \( (\frac{ae/b}{d}, \frac{cf/d}{d}) \) gets arbitrarily big as \( f \) gets big. The idea that the interval \( f(c/d - a/b) \) gets arbitrarily large for large values of \( f \) is called the Archimedean property of real numbers.

Thus, the solution to Problem 1 is to find the smallest positive integer \( f \) such that there exists a positive integer \( e \) such that \( e \) lies strictly between \( \frac{ae/b}{d} \) and \( \frac{cf/d}{d} \). Also, for this smallest positive integer \( f \), we choose \( e \) so that \( e \) is the smallest positive integer lying in \( (\frac{ae/b}{d}, \frac{cf/d}{d}) \). Thus, \( e \) will be the smallest positive integer greater than \( \frac{ae/b}{d} \). Thus, if \( \frac{ae/b}{d} \) is not an integer then \( \lceil \frac{ae/b}{d} \rceil = e \) and if \( \frac{ae/b}{d} \) is an integer then \( \lceil \frac{ae/b}{d} \rceil + 1 = \frac{ae/b}{d} + 1 = e \). We note that \( \frac{e}{f} \) is automatically reduced to lowest terms in the above solution.

A calculator can carry out the above solution.

Proof of Theorem 1. Suppose \( \frac{e}{f} \) is the solution to Problem 1 and \( \frac{\overline{e}}{\overline{f}} \) is the solution to Problem 1’ where \( \frac{e}{f}, \frac{\overline{e}}{\overline{f}} \) have the same values in both problems. We must show that \( (e, f, \frac{e}{f}) = (\overline{e}, \overline{f}, \frac{\overline{e}}{\overline{f}}) \). Now \( \frac{e}{f} \) solves \( 0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d} \). By Problem 1’, \( \overline{e} \) is the smallest positive integer such that a solution \( 0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d} \) exists. Therefore, (*) \( \overline{e} < e \) or \( \overline{e} = e \). Now \( \frac{\overline{e}}{\overline{f}} \) solves \( 0 \leq \frac{a}{b} < \frac{\overline{e}}{\overline{f}} < \frac{c}{d} \). By the
definition of \( f \) in Problem 1, we have \( f \leq \bar{f} \). Suppose \( \bar{e} < e \). If \( \bar{e} < e \), we know that \( f \leq \bar{f} \). Also, \( \bar{f} = f \) is impossible by the definition of \( f \) and \( e \) in Problem 1. Therefore, \( f < \bar{f} \).

Now \( 0 \leq \frac{a}{b} < \frac{\bar{e}}{\bar{d}} \) implies \( 0 \leq \frac{a}{b} < \bar{e} < \frac{\bar{a}}{\bar{d}} \).

Also, \( 0 \leq \frac{a}{b} < \frac{\bar{e}}{\bar{d}} \) implies \( 0 \leq \frac{a}{b} < e < \frac{\bar{a}}{\bar{d}} \). From the solution to Problem 1, we know that \( e \) is the smallest integer greater than \( \frac{a}{b} \). Now \( f < \bar{f} \) implies \( \frac{a}{b} < \frac{\bar{a}}{\bar{d}} \) if \( \frac{a}{b} \neq 0 \) or \( \frac{a}{b} = \frac{\bar{a}}{\bar{d}} \) if \( \frac{a}{b} = 0 \).

Thus, \( \frac{a}{b} < \frac{\bar{a}}{\bar{d}} < \bar{e} \).

We are supposing \( \bar{e} < e \).

Now \( e \) is the smallest integer greater than \( \frac{a}{b} \) and \( \frac{a}{b} < \frac{\bar{a}}{\bar{d}} < \bar{e} \) implies \( e < \bar{e} \). Therefore, from (*) we have \( e = \bar{e} \). By definition of \( f \), we have \( f < \bar{f} \). Since \( e = \bar{e} \), the smallest integer \( \bar{f} \) that can go with \( e = \bar{e} \) so that

\[
0 \leq \frac{a}{b} < \frac{\bar{e}}{\bar{d}} \quad \text{is} \quad \bar{f} = f.
\]

Therefore, \( (e, f, \frac{\bar{e}}{\bar{d}}) = (\bar{e}, f, \frac{\bar{e}}{\bar{d}}) \).

If \( 0 < \frac{\bar{e}}{\bar{d}} \) is a positive fraction, reduced to lowest terms, let us call \( x + y \) the size of \( \frac{\bar{e}}{\bar{d}} \).

From Theorem 1, we know that the solution \( 0 \leq \frac{a}{b} < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} \) that we gave to Problem 1 will also be the fraction \( \frac{\bar{e}}{\bar{d}} \) (where \( \frac{\bar{e}}{\bar{d}} \) is reduced to lowest terms) such that \( \frac{\bar{e}}{\bar{d}} \) is the fraction of the smallest possible size \( e + f \) that lies strictly between \( \frac{a}{b}, \frac{\bar{e}}{\bar{d}} \).

Thus, if \( 0 \leq \frac{a}{b} < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} \), from Theorem 1, the size \( e + f \) of \( \frac{\bar{e}}{\bar{d}} \) (reduced to lowest terms) has the smallest possible value if and only if both \( e \) and \( f \) have the smallest possible values.

In Section 11, we use this to generalize Problems 1, 1'.

**Example 1.** Find a fraction \( \frac{\bar{e}}{\bar{d}} \) (reduced to lowest terms) such that \( 0 < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} \) and such that \( f \) has the smallest possible value.

For this smallest possible value of \( f \), choose the smallest possible value of \( e \) that can go with \( f \) so that \( 0 < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} \). From Theorem 1, these values

\[
(e, f, \frac{\bar{e}}{\bar{d}})
\]

will also compute the fraction \( \frac{\bar{e}}{\bar{d}} \) (reduced to lowest terms) such that \( \frac{\bar{e}}{\bar{d}} \) lies in \( 0 < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} < \frac{\bar{e}}{\bar{d}} \) and also \( \frac{\bar{e}}{\bar{d}} \) has the smallest possible size \( e + f \).

**Solution.** We easily carry out the elementary solution with a calculator.

\[
\begin{align*}
1 \cdot \frac{\bar{e}}{\bar{d}} &= .7, & 1 \cdot \frac{\bar{e}}{\bar{d}} &= .7333333. \\
2 \cdot \frac{\bar{e}}{\bar{d}} &= 1.4, & 2 \cdot \frac{\bar{e}}{\bar{d}} &= 1.4666666. \\
3 \cdot \frac{\bar{e}}{\bar{d}} &= 2.1, & 3 \cdot \frac{\bar{e}}{\bar{d}} &= 2.1999999. \\
4 \cdot \bar{f} &= 2.8, & 4 \cdot \bar{f} &= 2.933333. \\
5 \cdot \bar{f} &= 3.5, & 5 \cdot \bar{f} &= 3.666666. \\
6 \cdot \bar{f} &= 4.2, & 6 \cdot \bar{f} &= 4.3999999. \\
7 \cdot \bar{f} &= 4.9, & 7 \cdot \bar{f} &= 5.133333. \\
\end{align*}
\]

Thus, \( f = 7 \) is smallest positive integer such that \( 7 \cdot \frac{\bar{e}}{\bar{d}} = 4.9 < e < 7 \cdot \frac{\bar{e}}{\bar{d}} = 5.133333 \).
Also, \( e = 5 \) is the smallest \( e \) that can go with \( f = 7 \) and the answer is \( \frac{e}{f} = \frac{5}{7} \).

In Section 9, we solve this example in two more (equivalent) ways. One way uses continued fractions.

\section*{6 Two Transformations, Translation and Inversion.}

As we explain in detail later, to solve Problem 1 in a more advanced way, we use the following Transformations 1, 2 to keep simplifying Problem 1 until it can be easily solved. Then we reverse ourselves and work back to the original solution. Continued fractions can be used to “keep the books” for us.

\textbf{Transformation 1, translation by an integer.} For fixed \( n \in \{1,2,3,\cdots\} \), consider the transformation \( t \rightarrow t + n \). For \( 0 \leq x < y \) and \( x,y \) are rational, this transformation maps the interval \( [x,y] \) to \( [x+n,y+n] \).

Suppose, \( \frac{e}{f} \in (x,y) \) and \( \frac{e'}{f'} \in (x,y) \) where \( 0 \leq x < y \) and \( x,y \) are rational and \( \frac{e}{f}, \frac{e'}{f'} \) are positive fractions reduced to lowest terms. Also, \( \frac{e}{f} \neq \frac{e'}{f'} \).

Suppose \( \frac{e}{f} \) is also the fraction in \( (x,y) \) having the smallest possible size \( e + f \). Thus, \( f \leq f', e \leq e' \) and at least one of \( f < f', e < e' \) is true. Consider the translation \( t \rightarrow t + n, [x,y] \rightarrow [x+n,y+n] \). Thus, \( \frac{e}{f} \rightarrow n + \frac{e}{f} = \frac{nf+e}{f} \) and \( \frac{e'}{f'} \rightarrow n + \frac{e'}{f'} = \frac{nf'+e'}{f'} \).

Of course, both \( \frac{nf+e}{f} \) and \( \frac{nf'+e'}{f'} \) are reduced to lowest terms since \( \frac{e}{f} \) and \( \frac{e'}{f'} \) are reduced to lowest terms.

Now size \( \left( \frac{nf+e}{f} \right) = (n+1) f + e < (n+1) f' + e' \) is true. Consider the translation \( t \rightarrow t + n, [x,y] \rightarrow [x+n,y+n] \), we see that the fraction \( \frac{e}{f} \) of the smallest possible size in \( (x,y) \) maps into the fraction \( n + \frac{e}{f} = \frac{nf+e}{f} \) in \( (x+n,y+n) \) that also has the smallest possible size.

Of course, from this, when we transform back again \( t \rightarrow t - n, [x+n,y+n] \rightarrow [x,y] \) we see that the fraction \( \frac{e}{f} \) in \( (x+n,y+n) \) having the smallest possible size will be mapped into the fraction \( \frac{e'}{f'} = \frac{e'}{f'} - n \) in \( (x,y) \) that also has the smallest possible size. Thus, the smallest size property is an invariant in the translations \( t \rightarrow t + n, [x,y] \rightarrow [x+n,y+n] \), \( t \rightarrow t - n, [x+n,y+n] \rightarrow [x,y] \).

We now call both \( t \rightarrow t + n, t \rightarrow t - n \) translation by an integer. This very important invariant helps to simplify and solve Problem 1. We show later that \( t \rightarrow t - n, [x+n,y+n] \rightarrow [x,y] \) simplifies Problem 1.

\textbf{Transformation 2, inversion.} Consider the transformation \( t \rightarrow \frac{1}{t} \). For \( 0 < x < y \) and \( x,y \) rational, inversion maps the interval \( [x,y] \) to \( \left[ \frac{1}{y}, \frac{1}{x} \right] \) and maps
the interval \([\frac{1}{y}, \frac{1}{x}]\) to \([x, y]\).

Now if \(0 < x < \frac{c}{f} < y\) where \(x, y\) are rational and \(\frac{c}{f}\) is a fraction reduced to lowest terms, we see that \(t \rightarrow \frac{1}{t}\) maps \(\frac{c}{f} \rightarrow \frac{f}{c}\). Of course, the size of \(\frac{c}{f}\) is \(e + f\) and the size of \(\frac{f}{c}\) is \(e + f\). Thus, the size of fractions (reduced to lowest terms) is invariant under \(t \rightarrow \frac{1}{t}\).

Also, the smallest size fraction \(\frac{c}{f}\) in \((x, y)\) maps into the smallest size fraction \(\frac{f}{c}\) in \((\frac{1}{y}, \frac{1}{x})\) and vice-versa.

**Observation 1** The elementary solution to Problem 1 is often much easier to carry out with a calculator if we first use inversion. We then solve Problem 1 for this new interval and then we transform back again.

7 Easy Standard Solutions to Problem 1

In Problem 1, the following cases 1-5 for the intervals \([\frac{a}{b}, \frac{c}{d}]\), \(0 \leq \frac{a}{b} < \frac{c}{d}\), with \(0 \leq \frac{a}{b} < \frac{c}{d}\), and considered easy and standard when computing the smallest size \(\frac{c}{f}\) in \((\frac{a}{b}, \frac{c}{d})\).

In all cases 1-5, \(n, m \in \{0, 1, 2, 3, \cdots\}\) and \(0 < \epsilon < 1\), \(0 < \bar{\epsilon} < 1\) and \(\epsilon, \bar{\epsilon}\) are rational. Also in each case, we list other restrictions on \(n, m, \epsilon, \bar{\epsilon}\) when more restrictions are needed. We give the solution for the smallest size \(\frac{c}{f}\) in each case 1-5. The proof of Case 3 is given in Section 10.

**Case 1.** \(\frac{c}{f} \in (n, m)\), \(0 \leq n < m\).

If \(n + 2 \leq m\) we have \(\frac{c}{f} = n + 1\).

If \(m = n + 1\), we have \(\frac{c}{f} = n + \frac{1}{2} = \frac{2n+1}{2}\).

**Case 2.** \(\frac{c}{f} \in (n, m + \epsilon)\), \(0 \leq n < m, 0 < \epsilon < 1\).

\(\frac{c}{f} = n + 1\).

**Case 3.** \(\frac{c}{f} \in (n + \epsilon, m)\), \(0 \leq n < m, 0 < \epsilon < 1\).

If \(n + 2 \leq m\), we have \(\frac{c}{f} = n + 1\).

If \(n + 1 = m\), we have \(\frac{c}{f} = n + \frac{f-1}{f}\) where \(f\) is the smallest positive integer that satisfies \(\frac{c}{f} > \frac{1}{1-\epsilon}\).

We prove this Case 3 in Section 10.
Case 4. \( \frac{e}{f} \in (n, n + \bar{\epsilon}) \), \( 0 \leq n \), \( 0 < \bar{\epsilon} < 1 \). We have \( \frac{e}{f} = n + \frac{1}{f} \) where \( f \) is the smallest positive integer that satisfies \( f > \frac{1}{\bar{\epsilon}} \). To see this, note that \( \frac{e}{f} = n + \frac{\bar{e}}{f} < n + \bar{\epsilon} \) is equivalent to \( \frac{\bar{\epsilon}}{e} < f \). We use Theorem 1 in the following.

Note that \( f \) has the smallest possible value if and only if \( \frac{e}{f} = 1 \) and when \( \frac{e}{f} = 1 \), \( f \) is the smallest positive integer such that \( f > \frac{1}{\mathcal{e}} \).

Case 5. \( \frac{e}{f} \in (n + \epsilon, m + \epsilon) \), \( 0 \leq n, m \), \( 0 < \epsilon < 1 \), \( 0 < \bar{\epsilon} < 1 \), we have \( \frac{e}{f} = n + 1 \).

8 Two Hard Cases

Cases 6, 7 are the two hard cases when solving Problem 1.

Case 6. \( \frac{e}{f} \in (n + \epsilon, n + n + \epsilon) \), \( n \in \{1, 2, 3, \ldots\} \), \( 0 < \epsilon < 1 \), \( \epsilon, \bar{\epsilon} \) are rational.

Case 7. \( \frac{e}{f} \in (\epsilon, \epsilon) \), \( 0 < \epsilon < 1 \), \( \epsilon, \bar{\epsilon} \) are rational.

We solve Cases 6, 7 as follows.

In case 6, we use translation: \( t \rightarrow t - n \) to transform \( [n + \epsilon, n + \bar{\epsilon}] \rightarrow [\epsilon, \bar{\epsilon}] \). Translation transforms the smallest size \( \frac{e}{f} \in (n + \epsilon, n + \bar{\epsilon}) \) into the smallest size \( \frac{e}{f} \rightarrow \frac{e}{f} - n = \frac{e - fn}{f} \in (\epsilon, \bar{\epsilon}) \).

Also, note that the size of \( \frac{e}{f} \) is reduced in the translation \( \frac{e}{f} \rightarrow \frac{e}{f} - n = \frac{e - fn}{f} \) since \( e - fn + f < e + f \). It is of interest that the sizes of all rational numbers \( \frac{e}{f} \in (n + \epsilon, n + \bar{\epsilon}) \) are reduced in the translation \( t \rightarrow t - n \) including the reduction of the size of \( n + \epsilon \rightarrow \epsilon, n + \bar{\epsilon} \rightarrow \bar{\epsilon} \). However, it is the reduction in the size of the smallest size \( \frac{e}{f} \in (n + \epsilon, n + \bar{\epsilon}) \rightarrow \frac{e}{f} = \frac{e}{f} - n \in (\epsilon, \bar{\epsilon}) \) that is the most important. Of course, this reduction in the size of \( \frac{e}{f} \rightarrow \frac{e}{f} - n \) is simplifying the Problem 1.

In case 7, we use inversion \( t \rightarrow \frac{1}{t} \) to transform \( [\epsilon, \bar{\epsilon}] \rightarrow \left[\frac{1}{\bar{\epsilon}}, \frac{1}{\epsilon}\right] \). Note that \( 1 < \frac{1}{\epsilon} < \frac{1}{\bar{\epsilon}} \) since \( 0 < \epsilon < \bar{\epsilon} < 1 \). Of course, inversion transforms the smallest size \( \frac{e}{f} \in (\epsilon, \bar{\epsilon}) \) into the smallest size \( \frac{e}{f} \in \left(\frac{1}{\epsilon}, \frac{1}{\bar{\epsilon}}\right) \). The sizes of all fractions \( \frac{e}{f} \in (\epsilon, \bar{\epsilon}) \) are invariant under inversion \( \frac{e}{f} \rightarrow \frac{y}{x} \in \left(\frac{1}{\bar{\epsilon}}, \frac{1}{\epsilon}\right) \) since \( x + y = y + x \).

Now \( \frac{e}{f} \in \left(\frac{1}{\bar{\epsilon}}, \frac{1}{\epsilon}\right) \) must come under one of the Cases 1-6. We finish this analysis in Section 8.

9 Solving Problem 1

To solve Problem 1, we note that we can immediately solve the easy standard Cases 1-5. To handle Cases 6, 7 we use translation and inversion as we explained in
Section 7. In Case 6, we use translation $t \to t - n$ and in Case 7 we use inversion $t \to \frac{1}{t}$. When we use translation $t \to t - n$, $[n + \epsilon, n + \bar{\epsilon}] \to [\epsilon, \bar{\epsilon}]$, the sizes of all fractions $\frac{x}{y} \in [n + \epsilon, n + \bar{\epsilon}]$, $\frac{x}{y} \to \frac{x}{y} - n$, are reduced including a reduction in size of $n + \epsilon \to \epsilon, n + \bar{\epsilon} \to \bar{\epsilon}$, and a reduction in size of the smallest size $\frac{\epsilon}{\bar{\epsilon}} \in (n + \epsilon, n + \bar{\epsilon})$ where $\frac{\epsilon}{\bar{\epsilon}} - n$ and $\frac{\epsilon}{\bar{\epsilon}} - n \in (\epsilon, \bar{\epsilon})$. This reduction of the smallest size $\frac{\epsilon}{\bar{\epsilon}}$ is by far the most important.

Now Case 6 transformed into Case 7. However, Case 7 is transformed into one of the six cases 1-6. Now inversion $\frac{\epsilon}{\bar{\epsilon}} \to \frac{\epsilon}{\bar{\epsilon}}$ leaves the sizes of the smallest sizes $\frac{\epsilon}{\bar{\epsilon}}$ and $\frac{\epsilon}{\bar{\epsilon}}$ invariant, while translation $\frac{\epsilon}{\bar{\epsilon}} \to \frac{\epsilon}{\bar{\epsilon}} - n$ reduces the size of the smallest size $\frac{\epsilon}{\bar{\epsilon}}$. If we keep transforming Cases 6, 7 back and forth to each other, we would eventually run into an impossibility since we cannot keep reducing the size of the smallest size $\frac{\epsilon}{\bar{\epsilon}}$ forever. Thus, as we transform Cases 6, 7 back and forth, eventually we must use inversion $t \to \frac{1}{t}$ to transform Case 7 into one of the easy standard Cases 1-5.

When we reach one of the easy standard Cases 1-5, we solve this easy case for the smallest size $\frac{\epsilon}{\bar{\epsilon}}$ that lies in the interval of that case. Then we reverse ourselves step by step and go back until we compute the smallest size $\frac{\epsilon}{\bar{\epsilon}} \in (\frac{a}{b}, \frac{c}{d})$ where $0 \leq \frac{a}{b} < \frac{c}{d}$ is given in Problem 1.

If we study continued fractions, we observe that translation $[n + \epsilon, n + \bar{\epsilon}] \to [\epsilon, \bar{\epsilon}]$ and inversion $[\epsilon, \bar{\epsilon}] \to [\frac{1}{\epsilon}, \frac{1}{\bar{\epsilon}}]$, are just computing the continued fraction expansions for the two end points $n + \epsilon \to \epsilon, \epsilon \to \frac{1}{\epsilon}$ and $n + \bar{\epsilon} \to \bar{\epsilon}, \bar{\epsilon} \to \frac{1}{\bar{\epsilon}}$. Thus, the continued fraction expansions of the two end points of our intervals provide a convenient bookkeeping scheme for keeping track of what we are computing. We can use continued fractions to compute Cases 6, 7 back and forth until we transform Case 7 into one of the easy standard Cases 1-5. We then put the answer to the easy Case 1-5 into the continued fractions and use the continued fraction to compute the final answer $\frac{\epsilon}{\bar{\epsilon}}$ to the original Problem 1 where $0 \leq \frac{a}{b} < \frac{c}{d}$.

All of this will become clear in Section 9 when we work through four examples. In the first example of Section 9, we first work the example directly without using continued fractions. Then we work through the entire example using continued fractions. This will show the reader step by step exactly how the continued fraction method is carried out. We emphasize that the continued fraction method is just a convenient bookkeeping scheme for keeping track of our calculations.

10 Four Examples for Solving Problem 1

We now go through four examples of finding the smallest size $\frac{\epsilon}{\bar{\epsilon}} \in (\frac{a}{b}, \frac{c}{d})$, $0 \leq \frac{a}{b} < \frac{c}{d}$. In the first example, we first solve the Problem 1 directly by using transformation 1, 2. We transform Cases 6, 7 back and forth until we run into one of the easy Cases 1-5. Then we solve this easy case. Then we use the answer to this easy case to work
backwards to compute the original answer \( \xi \) to our Problem 1.

Then we show how these same direct calculations can also be carried out with continued fractions. This parallel work will show the reader exactly what is going on. As stated before, transforming Cases 6, 7 back and forth by using translation and inversion creates a continued fraction of \( \frac{a}{b}, \frac{c}{d} \). When we run into one of the easy Cases 1-5, we substitute the answer to this easy case directly into the continued fraction of either \( \frac{a}{b} \) or \( \frac{c}{d} \). Then we use the continued fractions to compute the answer \( \frac{e}{f} \) to our original Problem 1. The examples will make this clear to the reader.

**Example 1.** Find the smallest size \( \frac{e}{f} \) that lies on \( 0 < \frac{7}{10} < \frac{\xi}{\eta} < \frac{11}{15} \).

Note that Example 1 was also solved in Section 4.

**Solution 1.** We transform Cases 7, 6 back and forth until we run into one of the easy Cases 1-5. Then we work backwards to compute \( 0 < \frac{7}{10} < \frac{\xi}{\eta} < \frac{11}{15} \). \( t \rightarrow \frac{1}{t} : \left[ \frac{7}{10}, \frac{\xi}{\eta}, \frac{11}{15} \right] \rightarrow \left[ \frac{15}{11}, \frac{\xi}{\eta}, \frac{10}{7} \right] = \left[ 1 + \frac{4}{11}, \frac{\xi}{\eta}, 1 + \frac{3}{7} \right] \). Note that \( \frac{\xi}{\eta} \) maps into \( \frac{\xi}{\eta} = \frac{\xi}{\eta} - 1 \). Note that \( \frac{\xi}{\eta} \) maps into \( \frac{3}{7} = \frac{\xi}{\eta} - 1 \) where \( \frac{\xi}{\eta} \) is the smallest size fraction in \( \left( \frac{4}{11}, \frac{3}{7} \right) \). \( t \rightarrow \frac{1}{t} : \left[ \frac{4}{11}, \frac{\xi}{\eta}, \frac{3}{7} \right] \rightarrow \left[ \frac{7}{3}, \frac{\xi}{\eta}, \frac{11}{4} \right] \rightarrow \left[ 2 + \frac{1}{3}, \frac{\xi}{\eta}, 2 + \frac{3}{4} \right] \). Note that \( \frac{\xi}{\eta} \) maps into \( \frac{\xi}{\eta} = \frac{\xi}{\eta} - 2 \). Note that \( \frac{\xi}{\eta} \) maps into \( \frac{7}{10} = \frac{\xi}{\eta} - 2 \). \( t \rightarrow \frac{1}{t} : \left[ \frac{7}{10}, \frac{\xi}{\eta}, \frac{3}{4} \right] \rightarrow \left[ \frac{1}{4}, \frac{\xi}{\eta}, 3 \right] = \left[ 1 + \frac{1}{3}, \frac{\xi}{\eta}, 3 \right] \). This is Case 3. Of course, \( \frac{7}{10} \) is the smallest size fraction in \( \left( 1 + \frac{1}{3}, 3 \right) \).

From Case 3, \( \frac{7}{10} = 2 \). Working backwards, we have the following.

\[
\frac{\xi}{\eta} = \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}.
\]

\[
\frac{\xi}{\eta} = \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}, \quad \frac{\xi}{\eta} = \frac{1}{2}.
\]

\( \frac{\xi}{\eta} = \frac{\xi}{\eta} \) is the same answer that we computed in Section 3.

We note that the calculations that we just made are fairly hard to keep track of. We now use continued fractions to go through the same calculations in an easier and more compact way.

**Solution 2 (Continued Fractions)**

\[
\frac{7}{10} \rightarrow \frac{1}{1 + \frac{7}{10}} \rightarrow \frac{1}{1 + \frac{1}{1 + \frac{\xi}{\eta}}}.
\]
\[
\frac{11}{15} \rightarrow \frac{1}{1+\frac{1}{11}} \rightarrow \frac{1}{1+\frac{1}{2+\frac{1}{4}}} \rightarrow \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{3}}}}.
\]

Note that the numbers in these continued fractions are the same numbers that we computed in Solution 1. Especially note the end \((1 + \frac{1}{3}, 3)\).

We now deal with \(\frac{e}{f} \in (1 + \frac{1}{3}, 3)\) which is Case 3. \(\frac{e}{f} = 2\) is the smallest size fraction such that \(\frac{e}{f} \in (1 + \frac{1}{3}, 3)\). Substituting \(\frac{e}{f} = 2\) for either \(3\) or \(1 + \frac{1}{3}\), we compute \(\frac{e}{f} = \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{3}}}} = \frac{5}{7}\) which is the same answer that we computed in the first direct solution. The continued fractions contain exactly the same information as the direct solution. However, the information in the continued fractions is much more compact and easy to understand.

**Example 2.**
\[
0 < \frac{4}{77} < \frac{e}{f} < \frac{3}{56}.
\]
\[
\frac{4}{77} \rightarrow \frac{1}{19+\frac{3}{8}} \rightarrow \frac{1}{1+\frac{1}{1+\frac{1}{9+\frac{1}{3}}}}.
\]
\[
\frac{7}{65} \rightarrow \frac{1}{9+\frac{2}{7}} \rightarrow \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{3}}}}.
\]

We now deal with \(\frac{e}{f} \in (18 + \frac{2}{3}, 19 + \frac{1}{4})\) which is Case 5. The smallest size \(\frac{e}{f} \in (18 + \frac{2}{3}, 19 + \frac{1}{4})\) is \(\frac{e}{f} = 19\). We now substitute \(\frac{e}{f} = 19\) for \(19 + \frac{1}{4}\) or \(18 + \frac{2}{3}\). This gives us \(\frac{e}{f} = \frac{1}{10} = \frac{3}{25}\) which is the known answer.

**Example 3.**
\[
0 < \frac{8}{75} < \frac{e}{f} < \frac{7}{65}.
\]
\[
\frac{8}{75} \rightarrow \frac{1}{9+\frac{1}{8}} \rightarrow \frac{1}{1+\frac{1}{9+\frac{1}{3}}}.\]
\[
\frac{7}{65} \rightarrow \frac{1}{9+\frac{2}{7}} \rightarrow \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{3}}}}.
\]

We now deal with \(\frac{e}{f} \in (2 + \frac{2}{3}, 3 + \frac{1}{2})\) which is Case 5. The smallest size \(\frac{e}{f} \in (2 + \frac{2}{3}, 3 + \frac{1}{2})\) is \(\frac{e}{f} = 3\). We now substitute \(3\) for \(2 + \frac{2}{3}\) or \(3 + \frac{1}{2}\). This gives us \(\frac{1}{9+\frac{1}{4}} = \frac{3}{28}\) which is the known answer.

**Example 4.**
\[
0 < \frac{10}{259} < \frac{e}{f} < \frac{3}{77}.
\]
\[
\frac{10}{259} \rightarrow \frac{1}{25+\frac{1}{20}} \rightarrow \frac{1}{25+\frac{1}{1+\frac{1}{5}}}.
\]
\[
\frac{3}{77} \rightarrow \frac{1}{25+\frac{1}{5}} = \frac{1}{25+\frac{1}{1+\frac{1}{4}}}.
\]

We now deal with \(\frac{e}{f} \in (2, 9)\) which is Case 1. The smallest size \(\frac{e}{f} \in (2, 9)\) is \(\frac{e}{f} = 3\). We now substitute \(3\) for \(2\), or \(9\). This gives us \(\frac{e}{f} = \frac{1}{25+\frac{1}{1+\frac{1}{5}}} = \frac{1}{25+\frac{1}{1+\frac{1}{4}}} = \frac{4}{103}\) which is the known answer.
11 Solving Case 3 of Section 5

We need to find the smallest size \( \frac{e}{f} \in (n + \epsilon, n + 1) \), where \( n \in \{0, 1, 2, 3, \ldots\} \), \( 0 < \epsilon < 1 \) and \( \frac{e}{f} \) is reduced to lowest terms. Using Theorem 1, we can let \( \frac{e}{f} = n + \frac{\epsilon}{f} \) where \( 1 \leq \frac{\epsilon}{f} < \frac{1}{f} \) and \( \frac{\epsilon}{f} \) is reduced to lowest terms and \( \frac{f}{f} \in (\epsilon, 1) \) and \( \frac{f}{f} \) has the smallest possible size which is true if and only if both \( e, f \) have the smallest possible values. Now \( n + \frac{\epsilon}{f} \in (n + \epsilon, n + 1) \) if and only if \( (n + 1) - \left(n + \frac{\epsilon}{f}\right) < (n + 1) - (n + \epsilon) \).

That is, \( \frac{e}{f} < 1 - \epsilon \).

Now \( \frac{e}{f} < \frac{1}{f} \). If we observe that \( \frac{\epsilon}{f} > \frac{1}{f} + 1 \), we see that \( \frac{f}{f} \) has the smallest possible value if and only if we let \( e = \frac{f}{f} - 1 \) and \( \frac{f}{f} \) is defined as the smallest positive integer satisfying \( \frac{f}{f} > \frac{1}{f} \).

By Theorem 1, \( \frac{e}{f} = n + \frac{f-1}{f} \) must be the solution to Case 3.

12 Generalizations

The methods in this paper can be slightly modified to solve the same Problem 1 for the case where \( 0 \leq \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) and \( \frac{a}{b}, \frac{c}{d} \) are allowed to be irrational numbers.

Next, suppose \( x > 0, y > 0 \) are fixed. The solution in this paper is also computing the fraction \( \frac{e}{f} \) (reduced to lowest terms) such that \( 0 \leq \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) and such that \( xe + yf \) is minimized. This follows from Theorem 1. This can be generalized as follows.

A function \( F(x, y) : (0, \infty) \times (0, \infty) \rightarrow R \) is said to be strictly increasing if \( x \leq \overline{x}, y \leq \overline{y} \) and at least one of \( x < \overline{x}, y < \overline{y} \) implies \( F(x, y) < F(\overline{x}, \overline{y}) \).

The solution in this paper is also computing the fraction \( \frac{e}{f} \) (reduced to lowest terms) such that \( 0 \leq \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) and such that \( F(e, f) \) is minimized.

Examples of \( F(e, f) \) are \( F(e, f) = e \cdot f, F(e, f) = e^2 + 3f, F(e, f) = ef + f^2 + 4e, F(e, f) = e + f \).

Problems 1, 1’ and Theorem 1 can also be generalized as follows.

Suppose \( f(x) : (0, \infty) \rightarrow (0, \infty) \) and \( g(x) : (0, \infty) \rightarrow (0, \infty) \) are strictly increasing. Also, suppose \( F(x, y) : (0, \infty) \times (0, \infty) \rightarrow R \) is strictly increasing as defined above. We generalize Problems 1, 1’ and Theorem 1 as follows. We use the above \( f(x), g(x), F(x, y) \) in Problem 1, 1’ and in Theorem 1.

**Problem 1.** Suppose \( 0 < \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) and \( \frac{a}{b}, \frac{c}{d} \) are fractions that are reduced to lowest terms.

Find an ordered pair \([e, f]\), where \( e, f \in \{1, 2, 3, 4, \ldots\} \), such that \( 0 < \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d} \) and such that \( f \in \{1, 2, 3, \ldots\} \) has the smallest possible value. For this smallest possible value of \( f \), let us compute the smallest possible \( e \in \{1, 2, 3, \ldots\} \).
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that can go with \( f \) so that \( \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d} \). In Problem 1, we assume that there exists \( e', f' \in \{1, 2, 3, \cdots \} \) such that \( \frac{a}{b} < \frac{g(e')}{f(f')} < \frac{c}{d} \).

**Problem 1’** Suppose \( 0 < \frac{a}{b} < \frac{c}{d} \) where \( \frac{a}{b}, \frac{c}{d} \) are fractions that are reduced to lowest terms. Find an ordered pair \((e, f)\), where \( e, f \in \{1, 2, 3, \cdots \} \) such that \( \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d} \) and such that \( e \in \{1, 2, 3, \cdots \} \) has the smallest possible value. For this smallest possible value of \( e \), let us compute the smallest possible \( f \in \{1, 2, 3, \cdots \} \) that can go with \( e \) so that \( 0 < \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d} \). In Problem 1’, we assume that there exists \( e', f' \in \{1, 2, 3, \cdots \} \) so that \( 0 < \frac{a}{b} < \frac{g(e')}{f(f')} < \frac{c}{d} \).

**Theorem 1.** Suppose \( \frac{a}{b}, \frac{c}{d} \) have the same values in both Problems 1, 1’, then the solutions \((e, f)\) and \((e', f')\) in Problems 1, 1’ are the same. That is \((e, f) = (e', f')\).

The elementary solutions to the new Problems 1, 1’ are exactly the same as the elementary solutions to Problems 1, 1’ that are given in Section 4.

Also, the proof of the new Theorem 1 is almost exactly the same as the proof of Theorem 1 that is given in Section 4.

From Theorem 1, we know that the solution \((e, f) = (e, f)\) that we computed in Problems 1, 1’ is also computing the ordered pairs \(\{(e, f) : e, f \in \{1, 2, 3, \cdots \}\}\) such that \( F(e, f) \) is minimized where \( F(x, y) : (0, \infty) \times (0, \infty) \to R \) is a strictly increasing function and such that \( 0 < \frac{a}{b} < \frac{g(e)}{f(f)} < \frac{c}{d} \).

### 13 More Sample Problems

1. Suppose \( 0 < \frac{a}{b} < \frac{c}{d} \) where \( \frac{a}{b}, \frac{c}{d} \) are fractions that are reduced to lowest terms.

   Find an ordered pair \((e, f)\), where \( e, f \in \{1, 2, 3, \cdots \} \) such that \( 0 < \frac{a}{b} < \frac{e^2}{f} < \frac{c}{d} \) and such that \( e + f \) is minimized.

2. Suppose \( 0 < \frac{c}{f} \) is a fraction reduced to lowest terms. Find the smallest \( x \) satisfying \( 0 \leq x < \frac{c}{f} \) and the largest \( y \) satisfying \( \frac{c}{f} < y \) such that \((x, y)\) has the property that \( \frac{c}{f} \) is the smallest size fraction in \((x, y)\).

### 14 Rational Numbers.

The most common way to study rational numbers is to study them all at one time. Let’s begin. A rational number is a number which can be expressed as a ratio of two integers, \( a/b \) where \( b \neq 0 \). Let \( \mathbb{Q} \) denote the set of all rational numbers. That
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is,

\[ \mathbb{Q} = \{ x \mid x = a/b, a, b \in \mathbb{Z}, b \neq 0 \}, \]

where \( \mathbb{Z} \) denotes the set of integers. The following exercises will help you understand the structure of \( \mathbb{Q} \).

1. Prove that the set \( \mathbb{Q} \) is closed under addition. That is, prove that for any two rational numbers \( x = a/b \) and \( y = c/d \), \( x + y \) is a rational number.

2. Prove that the set \( \mathbb{Q} \) is closed under multiplication. That is, prove that for any two rational numbers \( x = a/b \) and \( y = c/d \), \( x \cdot y \) is a rational number.

3. Prove that the number midway between two rational numbers is rational.

4. For this essay, we assume the set \( \mathbb{R} \) of real numbers is the set of all positive and negative decimal numbers and the number zero. These decimals have three forms, those that terminate, i.e., have only finitely many non-zero digits, like 1.12500...; those that repeat like 1.3333... = 4/3, and those that do not repeat. Prove that all rational numbers of one of the first two types, and vice-versa, any number of the first two types is rational.

5. Let \( z \) be a positive irrational number. Prove that there is a positive rational \( r \) number less than \( z \).

6. Prove that the rational numbers \( \mathbb{Q} \) is dense in the set of real numbers \( \mathbb{R} \). That is, prove that between any two real numbers, there is a rational number.

In the following exercises and problems, we need the notion of unit fraction. A unit fraction is a fraction of the form \( 1/n \) where \( n \) is a positive integer. Thus, the unit fractions are \( 1/1, 1/2, 1/3, \ldots \).

1. **Fractions as Addresses** Divide the unit interval into \( n \)-ths and also into \( m \)-ths for selected, not too large, choices of \( n \) and \( m \), and then find the lengths of all the resulting subintervals. For example, for \( n = 2, m = 3 \), you get \( 1/3, 1/6, 1/6, 1/3 \). For \( n = 3, m = 4 \), you get \( 1/4, 1/12, 1/6, 1/6, 1/12, 1/3 \). Try this for \( n = 3 \) and \( m = 5 \). Can you find a finer subdivision into equal intervals that incorporates all the division points for both denominators? I got this problem from Roger Howe.

2. Here’s a problem from *Train Your Brain*, by George Grützer. ‘It is difficult to subtract fractions in your head’, said John. ‘That’s right’ said Peter, ‘but you know, there are several tricks that can help you. You often get fractions whose numerators are one less that their denominators, for instance,

\[
\frac{3}{4} - \frac{1}{2}.
\]
It’s easy to figure out the difference between two such fractions.

\[
\frac{3}{4} - \frac{1}{2} = \frac{4 - 2}{4 \times 2} = \frac{1}{4}.
\]

Another example is \(7/8 - 3/4 = (8 - 4)/(8 \cdot 4)\). ‘Simple, right?’ Can you always use this method?

3. Show that every unit fraction can be expressed as the sum of two different unit fractions.

4. Sums of unit fractions
   
   (a) Notice that \(2/7\) is expressible as the sum of two unit fractions: \(2/7 = 1/4 + 1/28\). But \(3/7\) cannot be so expressed. Show that \(3/7\) is not the sum of two unit fractions.
   
   (b) There is a conjecture of Erdös that every fraction \(4/n\) where \(n \geq 3\) can be written as the sum of three unit fractions with different denominators. Verify the Erdös conjecture for \(n = 23, 24, \) and \(25\).
   
   (c) Can you write \(1\) as a sum of different unit fractions all with odd denominators?
   
   (d) Can any rational number \(r, 0 < r < 1\) be represented as a sum of unit fractions?
   
   (e) Find all solutions to \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{3}{4}\) with \(a \leq b \leq c\).

5. In the Space Between
   
   (a) Name a fraction between \(1/2\) and \(2/3\). Give an argument that your fraction satisfies the condition.
   
   (b) Name a fraction between \(11/15\) and \(7/10\). How about between \(6/7\) and \(11/13\)?
   
   (c) Name the fraction with smallest denominator between \(11/15\) and \(7/10\). Or \(6/7\) and \(11/13\)?
   
   (d) First draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into \(7 + 13 = 20\) equal pieces. How many marks are on each piece?
   
   (e) A bicycle team of 7 people brings 6 water bottles, while another team of 13 people brings 11 water bottles. What happens when they share? Some of this material is from Josh Zucker’s notes on fractions taken from
a workshop for teachers at American Institute of Mathematics, summer 2009. Some of the material is from the book Algebra by Gelfand and Shen.

6. Let $x_1, x_2, \ldots, x_{12}$ be positive numbers. Show that at least one of the following statements is true:

$$\frac{x_1}{x_2} + \frac{x_3}{x_4} + \frac{x_5}{x_6} + \frac{x_7}{x_8} + \frac{x_9}{x_{10}} \geq 5 \quad \frac{x_{11}}{x_{12}} + \frac{x_2}{x_1} + \frac{x_4}{x_3} + \frac{x_6}{x_5} \geq 4 \quad \frac{x_8}{x_7} + \frac{x_{10}}{x_9} + \frac{x_{12}}{x_{11}} \geq 3$$

7. **Dividing Horses**

This problem comes from *Dude, Can You Count?*, by Christian Constanda. An old cowboy dies and his three sons are called to the attorney’s office for the reading of the will.

All I have in this world I leave to my three sons, and all I have is just a few horses. To my oldest son, who has been a great help to me and done a lot of hard work, I bequeath half my horses. To my second son, who has also been helpful but worked a little less, I bequeath a third of my horses, and to my youngest son, who likes drinking and womanizing and hasn’t helped me one bit, I leave one ninth of my horses. This is my last will and testament.

The sons go back to the corral and count the horses, wanting to divide them according to their pa’s exact wishes. But they run into trouble right away when they see that there are 17 horses in all and that they cannot do a proper division. The oldest son, who is entitled to half—that is $8\frac{1}{2}$ horses—wants to take 9. His brothers immediately protest and say that he cannot take more than that which he is entitled to, even if it means calling the butcher. Just as they are about to have a fight, a stranger rides up and agrees to help. They explain to him the problem. Then the stranger dismounts, lets his horse mingle with the others, and says “Now there are 18 horses in the corral, which is a much better number to split up. Your share is half” he says to the oldest son, “and your share is six”, he says to the second. “Now the third son can have one ninth of 18, which is two, and there is $18 - 9 - 6 - 2 = 1$ left over. The stranger gets on the 18th horse and rides away. How was this kind of division possible.

8. Consider the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{12}.$$ 

Find all the ordered pairs $(a, b)$ of real number solutions.
9. Suppose \( \{a, b, c, d\} = \{1, 2, 3, 4\} \).
   
   (a) What is the smallest possible value of
   \[
   a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}},
   \]
   
   (b) What is the largest possible value of
   \[
   a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}},
   \]

10. **Smallest Sum.** Using each of the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try \(99/96 + 97/98\) but that is not the smallest sum. This problem is due to Sam Vandervelt. Extend this problem as follows. Suppose \(0 < a < b < c < d\) are all integers. What is the smallest possible sum of two fractions that use each integer as a numerator or denominator? What is the largest such sum? What if we have six integers, \(0 < a < b < c < d < e < f\). Now here’s a sequence of easier problems that might help answer the ones above.
   
   (a) How many fractions \(a/b\) can be built with \(a, b \in \{1, 2, 3, 4\}\), and \(a \neq b\)?
   
   (b) How many of the fractions in (a) are less than 1?
   
   (c) What is the smallest number of the form \(\frac{a}{b} + \frac{c}{d}\), where \(\{a, b, c, d\} = \{1, 2, 3, 4\}\)?
   
   (d) What is the largest number of the form \(\frac{a}{b} + \frac{c}{d}\), where \(\{a, b, c, d\} = \{1, 2, 3, 4\}\)?

11. **Simpsons** (with thanks to http://www.cut-the-knot.com) Bart and Lisa shoot free throws in two practice sessions to see who gets to start in tonight’s game. Bart makes 5 out of 11 in the first session while Lisa makes 3 out of 7. Who has the better percentage? Is it possible that Bart shoots the better percentage again in the second session, yet overall Lisa has a higher percentage of made free throws? The answer is yes! This phenomenon is called Simpson’s Paradox.
   
   (a) Find a pair of fractions \(a/b\) for Bart and \(k/l\) for Lisa such that \(a/b > k/l\) and yet, Lisa’s percentage overall is better.

   The numbers 12/21 and 11/20 are called mediants. Specifically, given two fractions \(a/b\) and \(c/d\), where all of \(a, b, c,\) and \(d\) are positive integers, the mediant of \(a/b\) and \(c/d\) is the fraction \((a + c)/(b + d)\).
(b) Why is the mediant of two fractions always between them?

(c) Notice that the mediant of two fractions depends on the way they are represented and not just on their value. Explain Simpson’s Paradox in terms of mediants.

(d) Define the mediant $M$ of two fractions $a/b$ and $c/d$ with the notation $M(a/b, c/d)$. So $M(a/b, c/d) = (a + c)/(b + d)$. This operation is sometimes called ‘student addition’ because many students think this would be a good way to add fractions. Compute the mediants $M(1/3, 8/9)$ and $M(4/9, 2/2)$ and compare each mediant with the midpoint of the two fractions.

Now let’s see what the paradox looks like geometrically on the number line. Here, $B_1$ and $B_2$ represent Bart’s fractions, $L_1, L_2$ Lisa’s fractions, and $M_B, M_L$ the two mediants.

\[
\begin{array}{cccc}
L_1 & B_1 & M_B & M_L & L_2 & B_2 \\
\end{array}
\]

(e) (Bart wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $\frac{a}{b} < \frac{s}{t} < \frac{1}{2}$,

ii. $\frac{c}{d} < \frac{s}{t} < 1$, and

iii. $\frac{a}{b} < \frac{s}{t} < \frac{1}{2}$.

(f) (Lisa wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $\frac{a}{b} < \frac{s}{t} < \frac{1}{2}$,

ii. $\frac{u}{v} < \frac{c}{d} < 1$, and

iii. $\frac{a}{b} < \frac{s}{t} < \frac{1}{2}$.

12. Using the notation $M(a/b, c/d)$ we introduced above, write the meaning of each of the statements below and prove them or provide a counter example.

(a) The mediant operation is commutative.

(b) The mediant operation is associative.

(c) Multiplication distributes of ‘mediation’. 
13. The number of female employees in a company is more than 60% and less than 65% of the total respectively. Determine the minimum number of employees overall.

14. The fraction of female employees in a company is more than \( \frac{6}{11} \) and less than \( \frac{4}{7} \) of the total respectively. Determine the minimum number of employees overall.

15. The number of female employees in a company is more than 60% and less than 65% of the total respectively. Determine the minimum number of employees overall.

16. For positive integers \( m \) and \( n \), the decimal representation for the fraction \( \frac{m}{n} \) begins 0.711 followed by other digits. Find the least possible value for \( n \).

17. **Fabulous Fractions I**

   (a) Find two different decimal digits \( a, b \) so that \( \frac{a}{b} < 1 \) and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

   (b) Find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{b} + \frac{c}{d} < 1 \) and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

   (c) Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1 \) and the sum is as large as possible.

   (d) Find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{b} + \frac{c}{d} < 2 \) but is otherwise as large as possible. Prove that your answer is correct. Then change the 2 to 3 and to 4.

   (e) Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 2 \) and the sum is as large as possible. Then change the 2 to 3 and to 4.

   (f) Finally find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{b} + \frac{c}{d} > 1 \) but is otherwise as small as possible. Prove that your answer is correct. Then change the 1 to 2 and to 3.

18. Note that if \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \), then each positive integer \( a, b, c \) is no larger than 6. What is the largest integer \( d \) such that \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1 \) with \( a \leq b \leq c \leq d \)? What’s next?

19. **Fabulous Fractions II**

   (a) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to build some fractions whose sum is 1. For example \( \frac{3}{7} + \frac{8}{56} + \frac{21}{45} = \frac{3}{7} + \frac{1}{7} + \frac{3}{7} = 1 \). Find all solutions.
(b) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to fill in the boxes so that the arithmetic is correct.

\[
\square \times \square + \square \times \square + \square \times \square = 1.
\]

What is the largest of the three fractions?

20. Use exactly eight digits to form four two digit numbers \( ab, cd, ef, gh \) so that the sum \( \frac{ab}{cd} + \frac{ef}{gh} \) is as small as possible. As usual, interpret \( ab \) as \( 10a + b \), etc.

21. Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{b} + \frac{c}{d} = \frac{e}{f} \).

22. Notice that

\[ \frac{19}{95} = \frac{19}{95} = \frac{1}{5}. \]

Can you find more pairs of two-digit numbers, with the smaller one on top, so that cancellation of this type works? Do you have them all?

23. **Problems with Four Fractions.** These problems can be very tedious, with lots of checking required. They are not recommended for children.

(a) For each \( i = 1, 2, \ldots, 9 \), use all the digits except \( i \) to solve the equation

\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N
\]

for some integer \( N \). In other words arrange the eight digits so that the sum of the four fractions is a whole number. For example, when \( i = 8 \) we can write

\[
\frac{9}{1} + \frac{5}{2} + \frac{4}{3} + \frac{7}{6} = 14.
\]

(b) What is the smallest integer \( k \) such that \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k \)? Which digit is left out?

(c) What is the largest integer \( k \) such that \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k \)? Which digit is left out?

(d) For what \( i \) do we get the greatest number of integers \( N \) for which \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N \), where \( S_i = \{a, b, c, d, e, f, g, h\} \)?

(e) Consider the fractional part of the fractions. Each solution of \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N \) belongs to a class of solutions with the same set of fractional parts. For example, 5/2+8/4+7/6+3/9 = 6 and 5/1+7/6+4/8+3/9 = 7 both have fractional parts sets \{1/2, 1/3, 1/6\}. How many different fractional parts multisets are there?
(f) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = i \]
where each letter represents a different nonzero digit.

(g) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 2i \]
where each letter represents a different nonzero digit.

(h) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 3i \]
where each letter represents a different nonzero digit.

(i) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 4i \]
where each letter represents a different nonzero digit.

(j) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 5i \]
where each letter represents a different nonzero digit.

(k) Find the maximum integer value of
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i \]
where each letter represents a different nonzero digit.

24. Build a binary tree of fractions as follows. At the top, \( \frac{1}{1} \). Then below each fraction \( \frac{a}{b} \) there are two fractions, \( \frac{a}{a+b} \) and \( \frac{a+b}{b} \), as shown in the below. This graph is called the Stern-Brocot Tree.
At what level do we find the fraction $\frac{3}{17}$?

25. I got this from Art Benjamin’s Magic of Math book. Let $n$ be an integer between 10 and 90. What is $n/91$? Compute the decimal equivalent.