In recent years, mathematics educators have begun to realize that understanding fractions and fractional arithmetic is the gateway to advanced high school mathematics. Yet, US students continue to do poorly when ranked internationally on fraction arithmetic. For example, consider this problem the 1995 TIMSS Trends in International Mathematics and Science Study:

What is $\frac{3}{4} + \frac{8}{3} + \frac{11}{8}$? The options were
A. $\frac{22}{15}$ B. $\frac{43}{24}$ C. $\frac{91}{24}$ D. $\frac{115}{24}$

More than 42 percent of US eighth graders who worked this problem chose option A. This percent was exceeded only by England. Singapore, Japan, and Belgium all had fewer than 10 percent students with answer A.

Consider the following real situation. A man bought a TV set for $1000, only to return a week later, complaining to the manager that he had bought the sets only because of the advertisement that the price was set ‘at a fraction of the Manufacturers’ Suggested Retail Price (MSRP)’. But the man had seen on the internet that the MSRP was only $900. Well, the manager said, ‘ten ninths is a fraction.’ The story does not end there. The customer won a small claims decision because the judge ruled that the manager was a crook. The common understanding of fraction, the judge said, is a number less than 1.

In the words of comedian Red Skelton: Fractions speak louder than words.

What is important here is to note that CCSS requires a model for fraction that enables both conceptual understanding and computational facility. The suggested model is to first understand unit fractions in more of less the same way we understand place value numbers in building an understanding of decimal representation. A place value number (or special number), is simply a digit times a power of 10. These place value numbers are the atoms of the number system, and we learn how to build the other numbers from them and how these numbers enable computation. Now for unit fractions, we can say that every fraction (at the elementary stage we still do not make the distinction between fraction and rational number as we do below) is a sum of unit fractions:

$$\frac{m}{n} = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}.$$ 

This is similar to the idea that every positive integer is expressible as a sum of place value numbers. Once we learn to perform the arithmetic operations of place value numbers, we are led naturally to the general case. Such happens with fractions also. For a discussion of place value see the paper I co-authored with Roger Howe: http://www.teachersofindia.org/en/article/five-stages-place-value.

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Fraction versus rational number. What’s the difference? It’s not an easy question. In fact, the difference is somewhat like the difference between a set of words on one hand and a sentence on the other. A symbol is a fraction if it is written a certain way, but a symbol that represents a rational number is a rational number no matter how it is written. Here are some examples. The symbol \( \frac{1}{2} \) is a fraction that is not a rational number. On the other hand \( \frac{2}{3} \) is both a fraction and a rational number. Now 0.75 is a rational number that is not a fraction, so we have examples of each that is not the other. To get a little deeper, a fraction is a string of symbols that includes a fraction bar, a numerator and a denominator. These items may be algebraic expressions or literal numbers. Any real number can be written as a fraction (just divide by 1). But whether a number if rational depends on its value, not on the way it is written. What we’re saying is that in the case of fractions, we are dealing with a syntactic issue, and in case of rational numbers, a semantic issue, to borrow two terms from computer science. For completeness, we say that a number is rational if it CAN be represented as a quotient of two integers. So 0.75 is rational because we can find a pair of integers, 3 and 4, whose quotient is 0.75.

Here’s another way to think about the difference. Consider the question ‘Are these numbers getting bigger or smaller?’

\[
\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}
\]

This apparently amusing question can provoke some serious questions about what we mean by the word ‘number’. Indeed, there are two aspects of numbers that often get blurred together: the value of a number and the numeral we write for the number. By ‘number’ we usually mean the value while the word ‘numeral’ refers to the symbol we use to communicate the number. So the numbers above are getting smaller while the numerals are getting bigger. This contrast between symbol and substance also explains the difference between rational number and fraction. A fraction is a numeral while a rational number is a number.

Rational Numbers. The most common way to study rational numbers is to study them all at one time. Let’s begin. A rational number is a number which can be expressed as a ratio of two integers, \( a/b \) where \( b \neq 0 \). Let \( \mathbb{Q} \) denote the set of all rational numbers. That is,

\[
\mathbb{Q} = \{ x \mid x = a/b, a, b \in \mathbb{Z}, b \neq 0 \},
\]

where \( \mathbb{Z} \) denotes the set of integers. The following exercises will help you understand the structure of \( \mathbb{Q} \).

1. Prove that the set \( \mathbb{Q} \) is closed under addition. That is, prove that for any two rational numbers \( x = a/b \) and \( y = c/d \), \( x + y \) is a rational number.
Solution: We simply need to write $x + y$ as a ratio of two integers. Because of the way we add fractions,

\[
x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} = \frac{ad + bc}{bd}.
\]

But $ad + bc$ and $bd$ are both integers because the integers are closed under both $+$ and $\times$.

2. Prove that the set $\mathbb{Q}$ is closed under multiplication. That is, prove that for any two rational numbers $x = \frac{a}{b}$ and $y = \frac{c}{d}$, $x \cdot y$ is a rational number.

Solution: We simply need to write $x + y$ as a ratio of two integers. Because of the way we multiply fractions,

\[
x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}
\]

Of course, $ac$ and $bd$ are both integers because the integers are closed under $\times$.

3. Prove that the number midway between two rational numbers is rational.

Solution: Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$. The midpoint of $x$ and $y$ is

\[
\frac{x + y}{2} = \frac{ad + bc}{2bd},
\]

which is a quotient of two integers.

4. For this essay, we assume the set $\mathbb{R}$ of real numbers is the set of all positive and negative decimal numbers and the number zero. These decimals have three forms, those that terminate, i.e. have only finitely many non-zero digits, like 1.12500...; those that repeat like 1.3333... = 4/3, and those that do not repeat. Prove that all rational numbers of one of the first two types, and vice-versa, any number of the first two types is rational.

Solution: To see that each rational number has either a terminating or repeating decimal representation, suppose $x = \frac{a}{b}$ is rational with $a$ and $b$ integers. Dividing $a$ by $b$ using long division, results in a sequence of remainders, and
each of these remainders is between 0 and \( b - 1 \). If the remainder is ever zero, the division process terminates and the resulting decimal has only a finite number of non-zero digits. If not, then we eventually get a repeat remainder, and once this happens, the remainders reoccur in blocks. The proof in the other direction requires more care. See the essay on representation for this proof.

5. Let \( z \) be a positive irrational number. Prove that there is a positive rational number less than \( z \).

**Solution:** Since \( z \) is positive, we can write \( z = x_0.x_1x_2\ldots \), where either \( x_0 \) or one of the \( x_i \) is not zero. Let \( k \) denote the smallest index such that \( x_k \) is not zero. Then \( x_0.x_1x_2\ldots x_k \) is a rational number less than \( z \).

6. Prove that the rational numbers \( \mathbb{Q} \) is dense in the set of real numbers \( \mathbb{R} \). That is, prove that between any two real numbers, there is a rational number.

**Solution:** Suppose \( x \) an \( y \) are any two real numbers with \( x < y \). Let \( x = x_0.x_1x_2\ldots \), and let \( y = y_0.y_1y_2\ldots \). Suppose \( k \) is the smallest subscript where they differ. Then \( y_0.y_1y_2\ldots y_k \) is a rational number between \( x \) and \( y \).

In the following exercises and problems, we need the notion of unit fraction. A unit fraction is a fraction of the form \( \frac{1}{n} \) where \( n \) is a positive integer. Thus, the unit fractions are \( \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \).

1. **Fractions as Addresses** Divide the unit interval into \( n \)-ths and also into \( m \)-ths for selected, not too large, choices of \( n \) and \( m \), and then find the lengths of all the resulting subintervals. For example, for \( n = 2, m = 3 \), you get \( 1/3, 1/6, 1/6, 1/3 \). For \( n = 3, m = 4 \), you get \( 1/4, 1/12, 1/6, 1/6, 1/12, 1/3 \). Try this for \( n = 3 \) and \( m = 5 \). Can you find a finer subdivision into equal intervals that incorporates all the division points for both denominators? I got this problem from Roger Howe.

**Solution:** It is probably best to do after displaying and ascertaining some appreciation for the way the multiples of a fixed unit fraction divide the number ray into equal intervals. In contrast, when you superimpose two such divisions, the resulting intervals will be quite different in length, and the situation may seem chaotic. A take away would be, you can, and the LCM of the denominators will give such. Also, there will always be at least one interval whose length is \( 1/LCM \), and of course, all intervals will be multiples of \( 1/LCM \).

2. Here’s a problem from *Train Your Brain*, by George Grätzer. ‘It is difficult to subtract fractions in your head’, said John. ‘That’s right’ said Peter, ‘but
you know, there are several tricks that can help you. You often get fractions whose numerators are one less that their denominators, for instance,

\[ \frac{3}{4} - \frac{1}{2} \]

It’s easy to figure out the difference between two such fractions.

\[ \frac{3}{4} - \frac{1}{2} = \frac{4 - 2}{4 \times 2} = \frac{1}{4}. \]

Another example is \( \frac{7}{8} - \frac{3}{4} = \frac{(8 - 4)}{(8 \cdot 4)}. \) ‘Simple, right?’ Can you always use this method?

**Solution:** Yes, this always works. Let \( \frac{a}{a+1} \) and \( \frac{b}{b+1} \) be two such fractions with \( a < b \). The

\[ \frac{b}{b+1} - \frac{a}{a+1} = \frac{(a + 1)b - (b + 1)a}{(b+1)(a+1)} = \frac{b + 1 - (a + 1)}{(b+1)(a+1)}. \]

3. Show that every unit fraction can be expressed as the sum of two different unit fractions.

**Solution:** Note that \( \frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}. \)

4. **Sums of unit fractions**

(a) Notice that \( \frac{2}{7} \) is expressible as the sum of two unit fractions: \( 2/7 = \frac{1}{4} + \frac{1}{28}. \) But \( \frac{3}{7} \) cannot be so expressed. Show that \( \frac{3}{7} \) is not the sum of two unit fractions.

(b) There is a conjecture of Erdős that every fraction \( \frac{4}{n} \) where \( n \geq 3 \) can be written as the sum of three unit fractions with different denominators. Verify the Erdős conjecture for \( n = 23, 24, \) and \( 25. \)

**Solution:** \( \frac{4}{25} = \frac{1}{8} + \frac{1}{40} + \frac{1}{100}. \)

(c) Can you write \( 1 \) as a sum of different unit fractions all with odd denominators?

**Solution:** Yes, it can be done. One way is to add the reciprocals of each of the following: \( 3, 5, 7, 9, 11, 15, 21, 165, 693. \) I suspect there are lots of other ways also.

(d) Can any rational number \( r, 0 < r < 1 \) be represented as a sum of unit fractions?
**Solution:** You just use the greedy algorithm: subtract the largest unit fraction less than your current fraction. You can show that the numerator of the resulting fraction is always less than the numerator you started with, so the process converges.

(e) Find all solutions to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{3}{4}$ with $a \leq b \leq c$.

**Solution:** There are five solutions, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}); (\frac{1}{2}, \frac{1}{5}, \frac{1}{20}); (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}); (\frac{1}{2}, \frac{1}{8}, \frac{1}{8}); (\frac{1}{3}, \frac{1}{4}, \frac{1}{6})$. Can you prove that these are the only solutions?

5. **In the Space Between**

(a) Name a fraction between $\frac{1}{2}$ and $\frac{2}{3}$. Give an argument that your fraction satisfies the condition.

**Solution:**

$$\frac{1}{2} < \frac{7}{12} < \frac{2}{3}$$

Of course this is the midpoint or the average of the two fractions, $\frac{1}{2} (\frac{1}{2} + \frac{2}{3}) = \frac{1}{2} \cdot \frac{14}{12} = \frac{7}{12}$

$$\frac{1}{2} < \frac{5}{9} < \frac{2}{3}$$

This is the answer I got from a six year old. That he nailed the question did not surprise me, but his explanation did. He said, $\frac{1}{2}$ is the same as $\frac{5}{10}$ which is less that $\frac{5}{9}$. And $\frac{5}{9}$ is less that $\frac{6}{9}$ which is the same as $\frac{2}{3}$.

$$\frac{1}{2} < \frac{5}{8} < \frac{3}{3}$$

Who provides this answer? A carpenter or anyone who works with rules graduated in eighths of an inch.

$$\frac{1}{2} < \frac{3}{5} < \frac{2}{3}$$

My favorite answer is obtained by thinking about the water-sharing model. A pair of bicyclers has a bottle of water between them. Three other bicyclers have two bottles of water among them. If they all agree to meet and share equally, how much will each cycler get? This number $\frac{3}{5}$ is called the *mediant* of the two fractions. We see this answer a lot when fifth graders learn addition of fractions incorrectly. Let’s use the water sharing model to help our fifth graders understand fraction addition and fraction comparisons. By the way, do you ever wonder why 10 year old boys can very easily compare the fractions $\frac{6}{18}$ and $\frac{7}{19}$. Baseball!
(b) Name a fraction between \(11/15\) and \(7/10\). How about between \(6/7\) and \(11/13\)?

(c) Name the fraction with smallest denominator between \(11/15\) and \(7/10\). Or \(6/7\) and \(11/13\)?

(d) First draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into \(7 + 13 = 20\) equal pieces. How many marks are on each piece?

(e) A bicycle team of 7 people brings 6 water bottles, while another team of 13 people brings 11 water bottles. What happens when they share?

Some of this material is from Josh Zucker’s notes on fractions taken from a workshop for teachers at American Institute of Mathematics, summer 2009. Some of the material is from the book Algebra by Gelfand and Shen.

6. Let \(x_1, x_2, \ldots, x_{12}\) be positive numbers. Show that at least one of the following statements is true:

\[
\frac{x_1}{x_2} + \frac{x_3}{x_4} + \frac{x_5}{x_6} + \frac{x_7}{x_8} + \frac{x_9}{x_{10}} \geq 5 \quad \frac{x_{11}}{x_{12}} + \frac{x_2}{x_1} + \frac{x_4}{x_3} + \frac{x_6}{x_5} \geq 4 \quad \frac{x_8}{x_7} + \frac{x_{10}}{x_9} + \frac{x_{12}}{x_{11}} \geq 3
\]

**Solution:** If they are all false, then

\[
\frac{x_1}{x_2} + \frac{x_3}{x_4} + \frac{x_5}{x_6} + \frac{x_7}{x_8} + \frac{x_9}{x_{10}} + \frac{x_{11}}{x_{12}} + \frac{x_2}{x_1} + \frac{x_4}{x_3} + \frac{x_6}{x_5} + \frac{x_8}{x_7} + \frac{x_{10}}{x_9} + \frac{x_{12}}{x_{11}} < 12.
\]

But the fractions can be arranged in pairs of the form \(\frac{a}{b} + \frac{b}{a}\) and each of these sums is at least 2, which contradicts the inequality. Note that \((a - b)^2 = a^2 - 2ab + b^2 \geq 0\). It follows that \(\frac{a^2}{ab} + \frac{b^2}{ab} \geq 2\).

7. **Dividing Horses**

This problem comes from *Dude, Can You Count?*, by Christian Constanda. An old cowboy dies and his three sons are called to the attorney’s office for the reading of the will.

All I have in this world I leave to my three sons, and all I have is just a few horses. To my oldest son, who has been a great help to me and done a lot of hard work, I bequeath half my horses. To my second son, who has also been helpful but worked a little less, I bequeath a third of my horses, and to my youngest son, who likes drinking and womanizing and hasn’t helped me one bit, I leave one ninth of my horses. This is my last will and testament.
The sons go back to the corral and count the horses, wanting to divide them according to their pa’s exact wishes. But they run into trouble right away when they see that there are 17 horses in all and that they cannot do a proper division. The oldest son, who is entitled to half—that is $8\frac{1}{2}$ horses—wants to take 9. His brothers immediately protest and say that he cannot take more than that which he is entitled to, even if it means calling the butcher. Just as they are about to have a fight, a stranger rides up and agrees to help. They explain to him the problem. Then the stranger dismounts, lets his horse mingle with the others, and says “Now there are 18 horses in the corral, which is a much better number to split up. Your share is half” he says to the oldest son, “and your share is six”, he says to the second. “Now the third son can have one ninth of 18, which is two, and there is $18 - 9 - 6 - 2 = 1$ left over. The stranger gets on the 18th horse and rides away. How was this kind of division possible.

**Solution:** The sum of the three fractions is less than 1: $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$. So the stranger’s horse helps complete the whole.

8. Consider the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{12}.$$ 

Find all the ordered pairs $(a, b)$ of real number solutions.

**Solution:** Thanks to Randy Harter for this problem. First, let’s try to find all integer solutions. To that end, rewrite the equation as

$$12(a + b) = 5ab.$$ 

Thus

$$0 = 5ab - 12b - 12a$$

$$= 5ab - 12b - 12a + \frac{144}{5} - \frac{144}{5}$$

$$= b(5a - 12) - \frac{12}{5}(5a - 12) - \frac{144}{5}$$

$$= (5a - 12)(5b - 12) - 144.$$ 

Now, factoring $144 = 2^43^2$ and looking at pairs of factors, we can find all the integral solutions. What about the rest. To this end, replace $a$ with $x$ and $b$ with $f(x)$ and solve for $f(x)$. We get

$$f(x) = \left( \frac{5}{12} - \frac{1}{x} \right)^{-1} = \frac{12x}{5x - 12}.$$
This rational function has a single vertical asymptote at \( x = \frac{12}{5} \) and a zero at \( x = 0 \). So you can see that for each \( 0 \neq x \neq \frac{12}{5} \), there is a \( y \) such that \( \frac{1}{x} + \frac{1}{y} = \frac{5}{12} \).

9. Suppose \( \{a, b, c, d\} = \{1, 2, 3, 4\} \).

(a) What is the smallest possible value of

\[
a + \frac{1}{\frac{b}{c} + \frac{1}{d}}.
\]

**Solution:** We can minimize the value by making the integer part as small as possible and then making the denominators as large as possible. Clearly \( a = 1 \) is the best we can do, and then \( b = 4 \) is certainly best, etc. So we get

\[
1 + \frac{1}{\frac{4}{2} + \frac{1}{3}} = \frac{38}{31}.
\]

(b) What is the largest possible value of

\[
a + \frac{1}{\frac{b}{c} + \frac{1}{d}}.
\]

**Solution:** We can maximize the value by making the integer part as large as possible and then making the denominators as small as possible. Clearly \( a = 4 \) is the best we can do, and then \( b = 1 \) is certainly best, etc. So we get

\[
4 + \frac{1}{\frac{1}{3} + \frac{1}{3}} = \frac{43}{9}.
\]

10. Smallest Sum. Using each of the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try \( \frac{99}{96} + \frac{97}{98} \) but that is not the smallest sum. This problem is due to Sam Vandervelt. Extend this problem as follows. Suppose \( 0 < a < b < c < d \) are all integers. What is the smallest possible sum of two fractions that use each integer as a numerator or denominator? What is the largest such sum? What if we have six integers, \( 0 < a < b < c < d < e < f \). Now here’s a sequence of easier problems that might help answer the ones above.

(a) How many fractions \( \frac{a}{b} \) can be built with \( a, b \in \{1, 2, 3, 4\} \), and \( a \neq b? \)

**Solution:** There are 12 fractions.
(b) How many of the fractions in (a) are less than 1?

**Solution:** There are 6 in this set, 1/2, 1/3, 1/4, 2/3, 2/4, 3/4.

(c) What is the smallest number of the form \( a/b + c/d \), where \( \{a, b, c, d\} = \{1, 2, 3, 4\} \)?

**Solution:** There are just three pairs of fractions to consider, \( A = \frac{1}{3} + \frac{2}{4} \), \( B = \frac{1}{4} + \frac{2}{3} \), and \( C = \frac{1}{2} + \frac{3}{4} \). Why is \( C \) not a candidate for the least value? Note that \( A \) beats \( B \) here because \( A = \frac{1}{3} + \frac{2}{4} = \frac{1}{3} + \frac{1}{4} \) while \( B + \frac{1}{4} + \frac{2}{3} = \frac{1}{4} + \frac{1}{3} + \frac{1}{3} \). Does this reasoning work for any four positive integers \( a < b < c < d \)? The argument for four arbitrary positive integers is about the same. Let \( A = \frac{a}{c} + \frac{b}{d} \), \( B = \frac{a}{d} + \frac{b}{c} \). Then \( A = \frac{a}{c} + \frac{b-a}{d} \), while \( B = \frac{a}{c} + \frac{b-a}{d} \), and since \( c < d \), \( A < B \).

(d) What is the largest number of the form \( a/b + c/d \), where \( \{a, b, c, d\} = \{1, 2, 3, 4\} \)?

**Solution:** Again there are just three pairs of fractions to consider, \( A = \frac{3}{1} + \frac{4}{2} \), \( B = \frac{4}{1} + \frac{3}{2} \), and \( C = \frac{2}{1} + \frac{4}{3} \). Why is \( C \) not a candidate for the least value? Note that \( B \) beats \( A \). Does this reasoning work for any four positive integers \( a < b < c < d \)?

11. **Simpson’s (with thanks to http://www.cut-the-knot.com)** Bart and Lisa shoot free throws in two practice sessions to see who gets to start in tonight’s game. Bart makes 5 out of 11 in the first session while Lisa makes 3 out of 7. Who has the better percentage? Is it possible that Bart shoots the better percentage again in the second session, yet overall Lisa has a higher percentage of made free throws? The answer is yes! This phenomenon is called Simpson’s Paradox.

(a) Find a pair of fractions \( a/b \) for Bart and \( k/l \) for Lisa such that \( a/b > k/l \) and yet, Lisa’s percentage overall is better.

**Solution:** One solution is that Bart makes 6 out of 9 in the second session while Lisa makes 9 out of 14. Now \( 12/21 > 11/20 \).

The numbers 12/21 and 11/20 are called mediants. Specifically, given two fractions \( a/b \) and \( c/d \), where all of \( a, b, c, \) and \( d \) are positive integers, the mediant of \( a/b \) and \( c/d \) is the fraction \( (a+c)/(b+d) \).

(b) Why is the mediant of two fractions always between them?

**Solution:** Imagine two groups of bicyclists, one with \( b \) riders and \( a \) bottles of water among them, and the other with \( d \) riders and \( c \) bottles of water among them. When they meet, they all agree to share equally. Thus we have \( b + d \) cyclists with a total of \( a + c \) bottles of water. Is it clear that \( \frac{a+c}{b+d} \) lies between \( a/b \) and \( c/d \)?
(c) Notice that the mediant of two fractions depends on the way they are represented and not just on their value. Explain Simpson’s Paradox in terms of mediants.

(d) Define the mediant $M$ of two fractions $a/b$ and $c/d$ with the notation $M(a/b, c/d)$. So $M(a/b, c/d) = (a + c)/(b + d)$. This operation is sometimes called ‘student addition’ because many students think this would be a good way to add fractions. Compute the mediants $M(1/3, 8/9)$ and $M(4/9, 2/2)$ and compare each mediant with the midpoint of the two fractions.

Now let’s see what the paradox looks like geometrically on the number line. Here, $B_1$ and $B_2$ represent Bart’s fractions, $L_1, L_2$ Lisa’s fractions, and $M_B, M_L$ the two mediants.

$$\begin{array}{cccccc}
L_1 & B_1 & M_B & M_L & L_2 & B_2 \\
\end{array}$$

(e) (Bart wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $a/b < s/t < 1/2$,

ii. $s/t < u/v < 1$,

iii. $a/c + c/d < s/u + u/v$.

(f) (Lisa wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $a/b < s/t < 1/2$,

ii. $s/t < u/v < 1$,

iii. $a/c + c/d < s/u + u/v$.

12. Using the notation $M(a/b, c/d)$ we introduced above, write the meaning of each of the statements below and prove them or provide a counter example.

(a) The mediant operation is commutative.

Solution: Yes, and this follows from the commutativity of addition of integers.

(b) The mediant operation is associative.

Solution: Yes, and this follows from the associativity of addition of integers.
(c) Multiplication distributes of ‘mediation’.

**Solution:** This is also true. Using \( a/b \triangle c/d \) instead of \( M(a/b, c/d) \), we have

\[
\frac{a}{b} \triangle \left( \frac{c}{d} \triangle \frac{e}{f} \right) = \frac{a}{b} \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{ac + ae}{bd}.
\]

13. The number of female employees in a company is more than 60% and less than 65% of the total respectively. Determine the minimum number of employees overall.

**Solution:** The answer is 8, because the fraction \( \frac{5}{8} \) is between 60% = \( \frac{3}{5} \) and 65% = \( \frac{13}{20} \).

14. The fraction of female employees in a company is more than 6/11 and less than 4/7 of the total respectively. Determine the minimum number of employees overall.

**Solution:** The answer is 9. The mediant of 6/11 and 4/7 is 10/18, which reduces to 5/9.

15. The number of female employees in a company is more than 60% and less than 65% of the total respectively. Determine the minimum number of employees overall.

**Solution:** By trial and error, the minimum number is 8. There could be 5 females and 3 males. This is also true in case we know that the percentage is less than the Fibonacci fraction 13/21, which makes the problem a bit harder.

16. For positive integers \( m \) and \( n \), the decimal representation for the fraction \( \frac{m}{n} \) begins 0.711 followed by other digits. Find the least possible value for \( n \).

**Solution:**

17. **Fabulous Fractions I**

(a) Find two different decimal digits \( a, b \) so that \( \frac{a}{b} < 1 \) and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

**Solution:** \( \frac{8}{9} \). This is as large as can be. Why?

(b) Find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{b} + \frac{c}{d} < 1 \) and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

**Solution:** \( \frac{7}{8} + \frac{1}{9} = \frac{71}{72} \). This is as large as can be. Why?

(c) Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1 \) and the sum is as large as possible.

**Solution:** One is \( \frac{3}{7} + \frac{1}{8} + \frac{4}{9} = \frac{503}{504} \).
(d) Find four different decimal digits \(a, b, c, d\) so that \(\frac{a}{b} + \frac{c}{d} < 2\) but is otherwise as large as possible. Prove that your answer is correct. Then change the 2 to 3 and to 4.

**Solution:** The denominator cannot be 72 or 63. Why? Trying for a fraction of the form \(\frac{2n-1}{n}\), where \(n = 56\), we are lead to \(\frac{9}{8} + \frac{6}{7} = \frac{111}{56}\). Why?

(e) Next find six different decimal digits \(a, b, c, d, e, f\) so that \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 2\) and the sum is as large as possible. Then change the 2 to 3 and to 4.

(f) Finally find four different decimal digits \(a, b, c, d\) so that \(\frac{a}{b} + \frac{c}{d} > 1\) but is otherwise as small as possible. Prove that your answer is correct. Then change the 1 to 2 and to 3.

**Solution:**

18. **Fabulous Fractions II**

(a) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to build some fractions whose sum is 1. For example \(\frac{3}{7} + \frac{8}{56} + \frac{21}{49} = \frac{3}{7} + \frac{1}{7} + \frac{3}{7} = 1\). Find all solutions.

**Solution:** Think about the multiples of 9: 9, 18, 27, 36, 45, 54, 63, 72, 81. Note that they use the same pair of digits or have no digits in common. Thus, \(\frac{9}{18} = \frac{18}{36} = \frac{27}{54} = \frac{36}{72}\) all have value 1/2. Note also that there is an odd operation \(\delta\) that preserves fraction value: \(\frac{9}{18}\delta\frac{3}{6} = \frac{93}{186}\). This operator is the key to building solutions to the equation. More later on this.

(b) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to fill in the boxes so that the arithmetic is correct.

\[
\begin{array}{ccc}
\Box & \Box & \Box \\
\Box \times \Box + \Box \times \Box + \Box \times \Box &=& 1
\end{array}
\]

What is the largest of the three fractions?

**Solution:** We may assume that the second numerator is 5 and the third is 7. If either 5 or 7 is used in a denominator, it can never be neutralized. Since the least common multiple of the remaining numbers is 72, we can use 1/72 as the unit of measurement. Now one of the three fractions must be close to 1. This can only be 5/(2 \cdot 3) or 7/(2 \cdot 4). In the first case, we are short 12 units. Of this, 7 must come from the third fraction so that the other 5 must come from the first fraction. This is impossible because the first fraction has numerator 1 and 5 does not divide 72. In the second case, we are 9 units short. In this case, 5 must come from the second fraction and 4 must come from the third. This can be achieved as shown below.
19. Use exactly eight digits to form four two digit numbers $ab, cd, ef, gh$ so that the sum $\frac{ab}{cd} + \frac{ef}{gh}$ is as small as possible. As usual, interpret $ab$ as $10a + b$, etc.

**Solution:** The answer is $\frac{13}{87} + \frac{24}{96}$. First, the four numerator digits are $1, 2, 3, 4$ and the four denominator digits are $6, 7, 8, 9$. Also, if $\frac{ab}{cd} + \frac{ef}{gh}$ is as small as possible, then $a < b$ and $c > d, e < f$ and $g > h$. For convenience, we assume $c < g$. Then $\frac{13}{87} + \frac{24}{96} = \frac{13}{13} + \frac{24}{96} - \frac{10}{96} > \frac{13}{87} + \frac{24}{96}$ because $\frac{10}{96} > \frac{1}{87}$.

Now compare $\frac{13}{87} + \frac{24}{96}$ with $\frac{13}{87} + \frac{24}{96}$. Pretty clearly, $\frac{13}{87} - \frac{13}{87} = \frac{13}{87} - \frac{13}{87} > 24\left(\frac{1}{96} - \frac{1}{96} = \frac{24}{96}\right) = \frac{24}{96}.$

20. Next find six different decimal digits $a, b, c, d, e, f$ so that $a + b = c + d$. $e + f$.

**Solution:** There are many solutions. One is $\frac{1}{3} + \frac{2}{4} = \frac{5}{6}$.

21. Notice that

$$\frac{19}{95} = \frac{19}{95} = \frac{1}{5}.$$

Can you find more pairs of two-digit numbers, with the smaller one on top, so that cancellation of this type works? Do you have them all?

**Solution:** Let us first build an equation using place value notation. Note that the equation can be written

$$\frac{10a + b}{10b + c} = \frac{a}{c},$$

where $a, b, c$ are digits and $a < c$. This leads to $c(10a + b) = a(10b + c)$, which we message to get $10ac - ac = 10ab - bc$. This, in turn leads to $9ac + bc = c(9a + b) = 10ab$. From this it follows that either $c = 5$ or $9a + b$ is a multiple of 5.

Case 1. $c = 5$. Then

$$\frac{10a + b}{10b + 5} = \frac{a}{5}.$$

Next, letting $a = 1, \ldots, a = 4$, etc. we find that

- $a = 1 : 5b + 50 = 10b + 5$ from which it follows that $b = 9$.
- $a = 2 : 100 + 5b = 20b + 10$ from which it follows that $90 = 15b$ and $b = 6$.
- $a = 3 : \frac{30 + b}{100 + 5} = \frac{3}{5}$ gives $150 + 5b = 30b + 15$ which has not integer solutions.
a = 4 : \[
\frac{40 + b}{10b + 5} = \frac{4}{5}
\]
gives rise to \(200 + 5b = 40b + 20\) which also has not solutions.

Why do we need not to check any higher values of \(a\)?

Case 2. \(9a + b\) is a multiple of 5. Again we consider \(a = 1, a = 2, \) etc.

\(a = 1 : 9 + b = 10\) or \(9 + b = 15\).If \(b = 9\) we get \(c = 5\) a case we already considered. If \(b = 6\) we get \(c = 4\), a new solution. \(ab = 16, bc = 64, \) and \(a/c = 1/4\).

\(a = 2 : 18 + b\) is either 20 or 25. One leads to \(b = 2\) and other \(b = 7\) and neither of these works.

\(a = 3 : \) This leads to \(b = 8\) which does not produce an integer value for \(c\).

\(a = 4 : 9 \cdot 4 + b = 45\), so \(b = 9\). This leads to \(c = 8\).

The four solutions are

\[
\begin{align*}
\frac{19}{95} &= \frac{19}{95} = \frac{1}{5} \\
\frac{26}{65} &= \frac{26}{65} = \frac{2}{5} \\
\frac{16}{64} &= \frac{16}{64} = \frac{1}{4} \\
\frac{49}{98} &= \frac{49}{98} = \frac{4}{8} = \frac{1}{2}.
\end{align*}
\]

22. **Problems with Four Fractions.** These problems can be very tedious, with lots of checking required. They are not recommended for children.

(a) For each \(i = 1, 2, \ldots, 9\), use all the digits except \(i\) to solve the equation

\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N
\]

for some integer \(N\). In other words arrange the eight digits so that the sum of the four fractions is a whole number. For example, when \(i = 8\) we can write

\[
\frac{9}{1} + \frac{5}{2} + \frac{4}{3} + \frac{7}{6} = 14.
\]

(b) What is the smallest integer \(k\) such that \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k\)? Which digit is left out?

**Solution:** The smallest achievable \(k\) is 2. It can be achieved when the digit left out is \(i = 5\) or \(i = 7\): \(1/4 + 7/6 + 2/8 + 3/9 = 5/4 + 1/6 + 2/8 + 3/9 = 2\).
(c) What is the largest integer $k$ such that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k$? Which digit is left out?

**Solution:** The largest achievable $k$ is 16. It can be achieved when the digit left out is $i = 4$ or $i = 5$: $9/1 + 7/2 + 8/3 + 5/6 = 8/1 + 7/2 + 9/3 + 6/4 = 9/1 + 8/2 + 7/3 + 4/6 = 16$.

(d) For what $i$ do we get the greatest number of integers $N$ for which $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N$, where $S_i = \{a, b, c, d, e, f, g, h\}$?

**Solution:** For $i = 5$ there are 36 solutions. For $i = 1, ..., 9$ we have $4, 3, 5, 4, 36, 3, 25, 7, and 21$ solutions respectively.

(e) Consider the fractional part of the fractions. Each solution of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N$ belongs to a class of solutions with the same set of fractional parts. For example, $5/2 + 8/4 + 7/6 + 3/9 = 6$ and $5/1 + 7/6 + 4/8 + 3/9 = 7$ both have fractional parts sets $\{1/2, 1/3, 1/6\}$. How many different fractional parts multisets are there?

**Solution:** There are 11 sets of fractional parts including the empty set. They are $\phi, \{1/2, 1/2\}, \{1/4, 3/4\}, \{1/3, 2/3\}, \{1/4, 1/4, 1/2\}, \{1/2, 3/4, 3/4\}, \{1/2, 1/3, 1/6\}, \{1/2, 2/3, 5/6\}, \{1/3, 1/2, 1/2, 2/3\}, \{1/6, 1/4, 1/4, 1/3\}, \{1/4, 1/3, 2/3, 3/4\}$.

(f) Find all solutions to $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = i$ where each letter represents a different nonzero digit.

**Solution:** There are two solutions with $i = 5$, one with $i = 7$, one with $i = 8$ and four with $i = 9$.

(g) Find all solutions to $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 2i$ where each letter represents a different nonzero digit.

**Solution:** There are two solutions with $i = 5$, one with $i = 6$, and three with $i = 7$.

(h) Find all solutions to $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 3i$ where each letter represents a different nonzero digit.

**Solution:** There is one solution with $i = 1$, one with $i = 4$, and four with $i = 5$.  

16
(i) Find all solutions to
\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 4i
\]
where each letter represents a different nonzero digit.

**Solution:** There are two solutions with \(i = 3\) and one with \(i = 4\).

(j) Find all solutions to
\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 5i
\]
where each letter represents a different nonzero digit.

**Solution:** There is only one solution: \(5/1 + 7/3 + 8/4 + 6/9 = 5 \cdot 2 = 10\).

(k) Find the maximum integer value of
\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i
\]
where each letter represents a different nonzero digit.

**Solution:** Let \(G_i\) denote the largest integer value of \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i\) where each letter represents a different nonzero digit. The largest possible value is 12. It is achieved when \(i = 4\); \(9/1 + 7/2 + 8/3 + 5/6 = 12\). The reasoning goes like this. The largest possible value of \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h}\) is \(9/1 + 8/2 + 7/3 + 6/4 = 16 + 2/3\), so the largest possible integer value is 16. This means that we need only check the values \(G_1, G_2,\) and \(G_3\). There are only four ways to get an integer by arranging the digits 2, 3, 4, 5, 6, 7, 8, 9 in the form \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h}\), and the resulting integers are 3, 6, 6, and 8. So \(G_1 = 8 - 1 = 7\). In case \(i = 2\) there are just three ways and the integers 7, 10, and 12 result, so \(G_2 = 12 - 2 = 10\). Finally, \(G_3\) is the largest of \(10 - 3, 12 - 3, 13 - 3,\) and \(14 - 3 = 11\). It would be enough to show that we cannot achieve 16 using all the nonzero digits except 3, 15 without the 2, or 14 without the 1, and this does not take too much work.