Place Value Problems

In this session, we’ll learn how to solve problems related to place value. This is one of the fundamental concepts in arithmetic, something every elementary and middle school mathematics teacher should understand profoundly. You already know that the digits used to represent a positive integer have different meanings depending on their position. This is commonly called place value.

Example 1. Pick a three digit number. Multiply it by 7. Then multiply your answer by 11, and finally multiply by 13. Explain why you got that answer.

Example 2. Next, consider the following problem. Find a four-digit number \(abcd\) which is reversed when multiplied by 9. In other words, find digits \(a, b, c,\) and \(d\) such that

\[
9 \cdot abcd = dcba.
\]

Solution. Our solution method is to reason digit by digit. First note that \(a = 1\) since otherwise \(9 \cdot abcd \geq 9 \cdot 2000 = 18000\), which is a five-digit number. Since \(a = 1\), it follows that \(d = 9\). Now the equation take the following form:

\[
9 \cdot 1bc9 = 9cb1.
\]
We can express this in decimal notation (in contrast to the underline notation we have been using) as follows:

\[
9 \cdot (1009 + 100b + 10c) = 9001 + 100c + 10b.
\]
Distributing the 9 across the \(\cdot\) yields

\[
9081 + 900b + 90c = 9001 + 100c + 10b
\]
from which it follows that

\[
80 + 890b = 10c.
\]
Since the right side 10c is at most 90 (\(c\) is a digit), we can conclude that \(b = 0\), and hence \(c = 8\). Therefore \(abcd = 1089\) is the only such number. In the exercises, you will be asked to investigate larger numbers that reverse when multiplied by 9. In the next example we deal with several digits at once.
Example 3. Find a 6-digit number \(abcdef\) that becomes 5 times as large when the units digit \(f\) is moved to the left end of the number. In other words, solve \(5 \cdot abcdef = fabcde\).

Solution. Before we solve this, let’s consider how a six-digit number changes when the rightmost digit is moved to the left end. Take 123456 as an example. Note that 612345 = 600000 + 12345, whereas 123456 = 123450 + 6 = 12345 \(\cdot\) 10 + 6. If we give the name \(x\) to 12345, a common technique in algebra, we can write 123456 = 10\(x\) + 6 and 612345 = 6 \(\cdot\) 10\(x\) + x. What the hypothesis tells us is that 5(10\(x\) + 6) = 6 \(\cdot\) 10\(x\) + x. Of course 123456 does not satisfy the equation, but replacing \(abcde\) with \(x\) reduces the six variables to just two. Of course we don’t know that \(f = 6\) works so we need to solve
\[
5 \cdot (10x + f) = f \cdot 10^5 + x.
\]
Distributing the 5 and migrating we get 50\(x\) + 5\(f\) = 10\(f\) + \(x\) which is equivalent to 49\(x\) = (10\(f\) - 5)\(f\) = 99995\(f\). Both sides are multiples of 7 so we can write 7\(x\) = 14285 \(\cdot\) \(f\). Now the left side is a multiple of 7, so the right side must also be a multiple of 7. Since 14285 is not a multiple of 7, it follows (from the Fundamental Theorem of Arithmetic, which we have yet to prove) that \(f\) must be a multiple of 7. Since \(f\) is a digit, it must be 7. And \(x\) must be 14285. This technique is one that you will use repeatedly in the next few weeks.

Example 4. The amazing number 1089. Start with your favorite two-digit number, reverse it and then subtract the smaller from the larger. What can you say about the result? Now pick a three-digit number like 742. Reverse it to get 247 and subtract the smaller of the two from the larger. Here we get 742 - 247 = 495. Now take the answer and reverse it to get 594, then add these two to get 495 + 594 = 1089. What did you get? Finally, compute the product of 1089 and 9. You get 1089 \(\cdot\) 9 = 9801. Isn’t that odd, multiplying by 9 had the effect of reversing the number. Is there a connection between these two properties?

Here’s a rationale for always getting 1089. First note, assuming \(a > c\) that \(abc - cba = 100(a - c) + c - a = 99(a - c) = 99d\). Now suppose 99\(d\) = \(u9v\). Why is the middle digit of 99\(d\) always a 9? Is there a connection between \(u\) and \(v\)? For \(d = 1\), we have \(u9v\) = 099, so \(u = 0\) and \(v = 9\). For \(d = 2\), \(u9v\) = 198. Again we see that \(u + v = 9\). Check out what we get for each of \(d = 3, \ldots, 9\). In each case \(u + v = 9\). Thus \(u9v + v9u = 100(u + v) + 18 + v + u = 900 + 18 + 9 = 1089\).
**Example 5.** Suppose \( a, b, c \) and \( d \) are digits (i.e., in the set \( \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \} \)), and the sum of the two four-digit numbers \( abcd \) and \( dabc \) is 6017. Find all four-digit numbers \( abcd \) with this property. Note that \( abcd \) is a four-digit number only if \( a \neq 0 \).

**Solution:** Let \( x \) be the three-digit number \( abc \). Then \( abcd = 10x + d \) and \( dabc = 1000d + x \), so the sum is \( 10x + d + 1000d + x = 11x + 1001d = 6017 \), which has lots of solutions. For \( d = 1 \), we get \( x = 456 \); \( d = 2 \) gives \( x = 365 \); \( d = 3 \) gives \( x = 274 \); and \( d = 4 \) gives \( x = 183 \). All the other values of \( d \) make \( x \) too small to be a 3-digit number.

Recall the two functions floor ([\( \lfloor \cdot \rfloor \)]) and fractional part (\( \langle \cdot \rangle \)), defined by \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \), and \( \langle x \rangle = x - \lfloor x \rfloor \). For example, \( \lfloor \pi \rfloor = 3 \) and \( \langle \pi \rangle = \pi - 3 \), \( \lfloor -\pi \rfloor = -4 \) and \( \langle -\pi \rangle = -\pi - (-4) = 4 - \pi \). Also, notice that for any real number \( x \), \( x = \lfloor x \rfloor + \langle x \rangle \). For example, \( \lfloor 3.15 \rfloor + \langle 3.15 \rangle = 3 + 0.15 = 3.15 \).

**Example 6.** Let \( N \) be a four-digit number and let \( N' = \langle N/10 \rangle \cdot 10^4 + [N/10] \). Suppose \( N - N' = 3105 \). Find all possible values of \( N \). Recall the two functions floor ([\( \lfloor \cdot \rfloor \)]) and fractional part (\( \langle \cdot \rangle \)), are defined by \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \), and \( \langle x \rangle = x - \lfloor x \rfloor \). Let \( N = abcd \). Then \( \langle N/10 \rangle = \langle abc.d \rangle = 0.d \), so 10000\( \langle N/10 \rangle \) could be written as 1000\( d \). On the other hand \( \lfloor N/10 \rfloor = \lfloor abc.d \rfloor = abc \). In other words, \( N' = dabc \), just as in the problem above. The equation \( 10x + d - (1000d + x) = 3105 \) is equivalent to \( 9x - 999d = 3105 \), and dividing both sides by 9 yields \( x - 111d = 345 \). There are six solutions for \( x \geq 100 \):

\[
\begin{align*}
d = 0 & \Rightarrow x = 345, N = 3450 \\
d = 1 & \Rightarrow x = 456, N = 4561 \\
d = 2 & \Rightarrow x = 567, N = 5672 \\
d = 3 & \Rightarrow x = 678, N = 6783 \\
d = 4 & \Rightarrow x = 789, N = 7894 \\
d = 5 & \Rightarrow x = 900, N = 9005
\end{align*}
\]
Example 7. Consider the equations

\[
\begin{align*}
1 \times 8 + 1 &= 9 \\
12 \times 8 + 2 &= 98 \\
123 \times 8 + 3 &= 987 \\
1234 \times 8 + 4 &= 9876
\end{align*}
\]

Find the next few equations in the list and prove that each one follows from the one above it.

Solution: Of course the next equations are

\[
\begin{align*}
12345 \times 8 + 5 &= 98765 \\
123456 \times 8 + 6 &= 987654 \\
1234567 \times 8 + 7 &= 9876543 \\
12345678 \times 8 + 8 &= 98765432 \\
123456789 \times 8 + 9 &= 987654321
\end{align*}
\]

Can you go further? Why does each one follow from the one before it? Start with the left side of the equation that has digit \(d\) in the units position. To help you understand, a string of digits like 123 is underlined, so that it is clear we are not multiplying.

\[
\begin{align*}
12\ldots d \times 8 + d &= (12\ldots (d - 1)0 + d) \times 8 + d \\
&= 12\ldots (d - 1)0 \times 8 + 8d + d \\
&= 10(12\ldots (d - 1) \times 8) + 9d \\
&= 10(12\ldots (d - 1) \times 8) + 10(d - 1) + (10 - d) \\
&= 10(12\ldots (d - 1) \times 8 + d - 1) + (10 - d) \\
&= 10(987\ldots (10 - d + 1) + (10 - d) \\
&= 987\ldots (10 - d + 1)0 + (10 - d) \\
&= 987\ldots (10 - d + 1)(10 - d)
\end{align*}
\]

Yes, you can go further. The next equation is

\[(1234567890 + 10) \times 8 + 10 = 9876543210.\]
During the rest of the session, please work on the following problems. You cannot finish them, but perhaps you will have time to work on them later.

1. **X’ing digits.** Consider the number

\[ N = 123456789101112 \ldots 5960, \]

obtained by writing the numbers from 1 to 60 in order next to one another. What is the largest number that can be produced by crossing out 100 digits of \( N \)? What is the smallest number that can be produced by crossing out 100 digits of \( N \)?

**Solution:** The number \( N \) has \( 9 + 2(51) = 111 \) digits, so the two answers will have to be 11 digit numbers. The largest could start with five 9s but not six (the sixth 9 is too close to the end). So we might try 9999978960. But we can do better, can’t we. How about 99999758596, which is bigger? The problem with this number is that the first 5 can be made larger simply by including the 0 at the end. Thus 99999785960 is the largest.

2. Use each of the five digits 1, 3, 5, 7 and 9 exactly once to build two numbers \( A \) and \( B \) such that \( A \cdot B \) is as large as possible. Then build two numbers \( C \) and \( D \) such that \( C \cdot D \) is as small as possible.

3. For each of the following problems, let \( S(n) = n \) in case \( n \) is a single digit integer. If \( n \geq 10 \) is an integer, \( S(n) \) is the sum of the digits of \( n \). Similarly \( P(n) \) is \( n \) if \( n \) is a positive single digit integer and the product of the digits of \( n \) otherwise. If there is no solution, prove it.

   (a) What is the smallest solution to \( S(n) = 2005 \). Express your answer in exponential notation.

   **Solution:** The smallest number with 222 nines and 1 seven, which is \( 8 \cdot 10^{222} - 1 \).

   (b) How many five-digit numbers \( n \) satisfy \( S(S(n)) + S(n) = 50 \).

   **Solution:** \( S(n) \) must be 43. Therefore \( n \) must have at least three 9’s and either two 8’s or another 9 and a 7. There are 10 five-digit numbers consisting of three 9’s and two 8’s and 5 five-digit numbers that have four 9’s and one 7, for a total of 15 numbers.
(c) Find all solutions to $S(S(n)) + S(S(n)) + S(n) = 100$.

**Solution:** There are no solutions because $S(S(n)) \equiv S(n) \equiv S(n) \pmod{9}$ but $100 \equiv 1 \pmod{9}$.

(d) Find all solutions to $S(S(n)) + S(n) + n = 2007$.

**Solution:** Since all three numbers are in the same congruence class modulo 9, and their sum is a multiple of 9, they must all be multiples of 3. If $n < 1969$, then $S(n) + S(S(n)) < 35$, so we need not consider any number smaller than 1969. If $1987 \leq n \leq 1999$, then $S(S(n)) + S(n) + n \geq 1990 + 19 = 2009$. Checking the other numbers in the range give the four solutions 1977, 1980, 1983, and 2001. This was problem 22 on the AMC12A in 2007.


**Solution:** Think of the first 2007 positive integers being written in a $2007 \times 4$ matrix, with first row 0001, second 0002, etc. Now compute the sums of the four columns. Starting with the left column, we have 999 zeros, 1000 ones, and 8 twos for a sum of 1016. In the hundreds position, we have 200 of each non-zero digit for a total of $200(1 + 2 + \cdots + 9) = 9000$. In the tens column, we have again 200 of each non-zero digit, another 9000. Then in the fourth column, the units digits, we have $200(45) + 1 + 2 + \cdots + 7 = 9028$. Now summing these four numbers give $1016 + 9000 + 9000 + 9028 = 28044$.

(f) Can both $S(a)$ and $S(a + 1)$ be multiples of 49?

(g) Find a number $n$ such that $S(n) = 2S(n + 1)$. For what values of $k$ does there exist $n$ such that $S(n) = kS(n + 1)$?

(h) Find, with proof, the largest $n$ for which $n = 7S(n)$.

**Solution:** This problem was inspired by a Problem of the Month proposal of Professor Bela Bajnok of Gettysburg College. First note that $n$ cannot be a four-digit or larger number. If so, we’d have $abcd = 7(a + b + c + d) \leq 7 \cdot 36 < 1000$, a contradiction. Now if $n = \overline{abc}$, and $n = 7S(n)$, then we have $100a + 10b + c = 7a + 7b + 7c$, which is equivalent to $93a + 3b = 6c$ and to $31a + b = 2c$. Only $a = 0$ can work here since $c$ is a digit. Solving $b = 2c$ for the largest possible values of $b$ and $c$ yields $b = 8, c = 4$, so $n = 84$ is the largest integer with the desired property.
(i) Find the smallest positive integer \( n \) satisfying \( S(S(n)) \geq 10 \).

(j) Find the smallest positive integer \( n \) satisfying \( S(S(n)) \geq 100 \).

(k) Find the smallest positive integer \( n \) such that \( S(n^2) = 27 \).

Solution: The only three-digit number with sum of digits 27 is 999, so we need only check values of \( n \) satisfying \( n \geq 32 \). Since 9 is a divisor of \( n \), 3 must be a divisor of \( n \). This means we need only check 33, 36, 39, . . . until we find a number whose square has sum of digits 27. This occurs at \( n = 63 \), and \( 63^2 = 3969 \).

4. Recall the two functions \( \text{floor} (\lfloor \rfloor) \) and \( \text{fractional part} (\langle \rangle) \), defined by \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \), and \( \langle x \rangle = x - \lfloor x \rfloor \).

(a) For each member \( x \) of the set \( S = \{\pi, 1.234, -1.234, \frac{7}{3}, -\frac{7}{3}\} \), evaluate \( \langle x \rangle \) and \( \lfloor x \rfloor \).

(b) Define another function \( f \) by \( f(x) = x - 10\lfloor \frac{x}{10} \rfloor \). Find \( f(x) \) and \( f(\lfloor x \rfloor) \) for each \( x \) in \( S \).

(c) Let \( g(x) = \lfloor \frac{x}{10} \rfloor - 10\lfloor \frac{x}{100} \rfloor \). Evaluate \( g \) at each of the members of \( S \).

(d) Prove that for any number \( x = 100a + 10b + c + f \), where \( a \) is a positive integer, \( b \) is a digit, \( c \) is a digit, and \( 0 \leq f < 1 \), \( g(x) = b \). In other words, \( g(x) \) is the tens digit of \( x \).

Solution: Note that

\[
\begin{align*}
g(x) &= \left\lfloor \frac{100a + 10b + c + f}{10} \right\rfloor - 10 \left\lfloor \frac{10a + 10b + c}{100} \right\rfloor \\
&= \left\lfloor \frac{100a + 10b + c}{10} \right\rfloor - 10 \left\lfloor \frac{10a + 10b + c}{100} \right\rfloor \\
&= \lfloor 10a + b + c/10 \rfloor - 10 \lfloor a + b/10 + c/100 \rfloor \\
&= 10a + b - 10a = b
\end{align*}
\]

5. Take the first three digits of your phone number (NOT the Area code...)
Then multiply by 80. Next add 1. Multiply by 250. Then add to this the last 4 digits of your phone number. Then add to this the last 4 digits of your phone number again. Subtract 250. Finally divide number by 2. What do you get and why does this work? Explain why you get such an interesting answer.
**Place Value Problems**

**Solution:** \[\frac{(80x + 1)(250) + 2y - 250}{2} = \frac{2000x + 2y}{2} = 1000x + y,\]
which is the phone number you entered. Alternatively, let \(abcdefg\) denote your seven-digit phone number and let \(x = abc\) and \(y = defg\). Then

\[
\frac{(80x + 1)250 + 2y - 250}{2} = \frac{20000x + 2y}{2} = 10000x + y = abc0000 + defg = abcdefg
\]

6. The numbers 1, 2, 3, 6, 7, 8 are arranged in a multiplication table, with three along the top and the other three down the column. The multiplication table is completed and the sum of the nine entries is tabulated. What is the largest possible sum obtainable.

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<thead>
<tr>
<th>(\times)</th>
<th>(a)</th>
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<tbody>
<tr>
<td>(d)</td>
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<td>(e)</td>
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**Solution:** The sum is \((a+b+c)\cdot(d+e+f)\) which is as large as possible when the two factors \(a+b+c\) and \(c+d+e\) are as close together as possible (given that the sum is constant(27)). So one is 13 and the other is 14 and the maximum sum is \(13\cdot14 = 182\).

7. A special-8 number is one whose decimal representation consists entirely of 0’s and 8’s. For example 0.8808 and 0.08 are special numbers. What is the fewest special numbers whose sum is 1.

**Solution:** 5. The five numbers that work are 0.888, 0.88, 0.008, 0.008, 0.008. To see that we cannot do better, divide all the numbers in the problem by 8, and try writing \(1/8 = 0.125\) as a sum of numbers all of whose digits are 0 or 1. Clearly we cannot do better than 5 of these.

8. A special-3 number is one whose decimal representation consists entirely of 0’s and 3’s. For example 0.3033 and 0.03 are special numbers. What is the fewest special numbers whose sum is 1.

**Solution:** 3. The three numbers that work are all the same, namely 0.3.
9. A special-7 number is one whose decimal representation consists entirely of 0’s and 7’s. For example 0.7707 and 0\,\overline{07} are special numbers. What is the fewest special numbers whose sum is 1.

**Solution:** 8. Writing 1 as a sum of special numbers is equivalent to writing \(\frac{1}{7}\) as a sum of numbers all of whose digits are 0 or 1. Since \(\frac{1}{7} = 0.1\overline{42857}\), it follows that we can do no better than 8 special numbers.

10. What is the largest 5-digit multiple of 11 that has exactly 3 different digits?

**Solution:** 99968. The largest must be a number of the form 999\,xy where \(x\) and \(y\) are suitable digits. To be a multiple of 11, the digits \(x\) and \(y\) must satisfy \(y + 9 + 9 - (x + 9) = y - x + 9\) is a multiple of 11. So \(y - x = 2\) and since neither \(x\) nor \(y\) can be 9, we have 99968.

11. A two-digit integer \(N\) that is not a multiple of 10 is \(k\) times the sum of its digits. The number formed by interchanging the digits is \(m\) times the sum of the digits. What is the relationship between \(m\) and \(k\)?

**Solution:** Note that \(10a + b = k(a + b)\) and \(10b + a = m(b + a)\). Adding the two equations together, it follows that \(k + m = 11\).

12. A check is written for \(x\) dollars and \(y\) cents, both \(x\) and \(y\) two-digit numbers. In error it is cashed for \(y\) dollars and \(x\) cents, the incorrect amount exceeding the correct amount by $17.82. Find a possible value for \(x\) and \(y\).

**Solution:** Note that \(100y + x - (100x + y) = 99(y - x) = 1782\). It follows that \(y - x = 18\), so any pair of two-digit numbers that differ by 18 will work.

13. Solve the alpha-numeric problem \(abcd \times 4 = dcba\), where \(a, b, c\) and \(d\) are decimal digits.

**Solution:** First note that if \(a \geq 3\) then \(4 \cdot abcd\) is a five-digit number. Thus \(a = 1\) or \(a = 2\). But since \(dcba\) is an even number, \(a = 2\). We have \(4(2000 + 100b + 10c + d) = 1000d + 100c + 10b + 2\). It follows quickly that \(c = d\). Then \(400b + (4c + 3)10 = 100c + 10b\). Thus \(39b + 3 = 6c\). Only \(c = 7\) works and it follows that \(b = 1\). To check, we have \(4 \cdot 2178 = 8712\).
14. The rightmost digit of a six-digit number $N$ is moved to the left end. The new number obtained is five times $N$. What is $N$?

**Solution:** The number is 142857. Let $N = \underline{abcdef}$ and let $x = \underline{abcde}$. Then we have $5N = 5(abcdef) = fabcde$. Note that we have adopted the convention of underlining the digits of an integer in decimal notation. For example $\underline{ab} = 10a + b$. Note that $5(abcdef) = 5(abcd0 + f) = 5(10x + f) = fabcde = 100000f + x$. This leads to $50x - x = 100000f - 5f$ or $49x = 99995f$. Factoring both sides gives $7^2x = 5 \cdot 7 \cdot 2857 \cdot f$, which is equivalent to $7x = 5 \cdot 2857 \cdot f$, from which it follows that $f = 7$. Therefore $x = 14285$ and $N = 142857$.

15. Repeat the same problem with the 5 changed to a 4. That is $4(abcdef) = \underline{fabcde}$

**Solution:** There are several solutions, $N = 128205$, $N = 153846$, $N = 102564$, $N = 205128$, $N = 179487$ and $N = 230769$.

16. Let $N = 1234567891011 \ldots 19992000$ be the integer obtained by appending the decimal representations of the numbers from 1 to 2000 together in order. What is the remainder when $N$ is divided by 9?

**Solution:** The answer is 3. We give two solutions. First we’ll compute the sum $S(N)$ of the digits of $N$. This can be done easily by thinking of the digits of $N$ as the entries of a 2000 by 4 matrix whose first row is 0001 and whose last row is 2000, with all the other integers from 1 to 2000 occupying a row. Then count the number of each nonzero digit. There are 1600 1’s, 601 2’s, and 600 of each of the other nonzero digits. This means that $S(N) = 600(1 + 2 + 3 + \cdots + 9) + 2 + 1000 = 28002$, and $28002 \equiv 3 \pmod{9}$.

The second solution is due to Mike Pillsbury, middle school teacher at Randolph Middle in Charlotte: $N \equiv S(N) = ((1 + 2000)/2) \cdot 2000 = 2001000 \equiv 3 \pmod{9}$. This works because the sum of the digits of $N$ is congruent to the sum of the numbers between 1 and 2000.

17. Ashley, Chris and Elizabeth have a large supply of identical cubical dice. Each one builds a pile of them directly upon one another. When they are done, they notice that:

(a) When they read the tops of the three columns from left to right as one number, the number is exactly the total number of dots
(b) When the top numbers are multiplied together, the product is exactly the number of dice used to build the three columns.

For example, if the top three dice looked like

![Die images](image.png)

we would be interpreted as the number 142. How many dice are there among the three piles?

**Solution:** First note that on a standard die, opposite numbers add to 7. That means that each die below the top contributes exactly 14 to the total. Let \(a, b, \text{ and } c\) denote the top numbers. From item 2, the number of dice is \(abc\). Then the total number of dots showing is \(14abc + a + b + c\) and this is the same as the number \(abc\). But \(abc\) is the number whose hundreds digit is \(a\), tens digit is \(b\) and units digit is \(c\). Thus we have

\[
14abc + a + b + c = 100a + 10b + c.
\]

This is equivalent to

\[
14abc = 99a + 9b = 9(11a + b).
\]

It follows from this that \(11a+b\) is a multiple of 14, and \(abc\) is a multiple of 9. To narrow this down a bit, \(14abc = 99a + 9b > 99a\), and so \(14bc > 99\). At least two of the digits \(a, b, c\) are either 3 or 6. Why? Call this condition \(*\). What are the possible values of \(bc\)? The answer is that \(8 \leq bc \leq 13\). We need only check \(bc = 9\) and \(bc = 12\) because of \(*\). If \(bc = 12\), then \(14a(12) = 991 + 9b\) and \(168a = 991 + 9b\), which is impossible. Now if \(bc = 9\) then it follows that \(b = c = 3\) and from this we learn that \(a = 1\), so our number is 133, there are 9 dice and the total number of dots showing is \(9(14) + 7 = 133\).

18. Find three three-digit numbers whose squares end with the digits 444.

**Solution:** They are 462, 538 and 962. Let \(k\) denote a number whose square ends in 444. Note that the units digit of \(k\) must be either 2 or 8
since any other units digit would yield a units digit of \( k^2 \) different from 4.

In case I, we consider those \( k \) for which the units digit is 2. Thus \( k = 10n + 2 \) and \( k^2 = 100n^2 + 40n + 4 \). In the language of modular congruences, this means 100n^2 + 40n + 4 \( \equiv \) 444 (mod 1000), and from this it follows that 100n^2 + 40n \( \equiv \) 440 (mod 1000). We can divide both sides by 10 to get 10n^2 + 4n \( \equiv \) 44 (mod 100), and this forces \( n \) to have a units digit of 1 or 6. Trying 6, we have \( k = 100t + 62 \) for some digit \( t \). Squaring both sides get \((100t + 62)^2 = 10000t^2 + 2 \cdot 6200t + 62^2 = 10000t + 12400t + 3844 = 1000w + 444 \) for some integer \( w \). Sorting this out, it follows that 4t + 8 \( \equiv \) 4 (mod 10), and this requires \( t = 4 \) or \( t = 9 \). They both work: 462^2 = 213444 and 962^2 = 925444. The other case, where the units digit of \( n \) is 1 does not lead to any solutions.

Now for case II. Thus \( k = 10n + 8 \) and \( k^2 = 100n^2 + 160n + 64 \). In the language of modular congruences, this means 100n^2 + 160n + 64 \( \equiv \) 444 (mod 1000), and from this it follows that 100n^2 + 160n \( \equiv \) 380 (mod 1000). We can divide both sides by 10 to get 10n^2 + 16n \( \equiv \) 38 (mod 100), and this forces \( n \) to have a units digit of 3 or 8. Trying 3, we have \( k = 100t + 38 \) for some digit \( t \). Squaring both sides get \((100t + 38)^2 = 10000t^2 + 2 \cdot 3800t + 38^2 = 10000t^2 + 7600t + 1444 = 1000w + 444 \) for some integer \( w \). Thus 100000t^2 + 7600t + 1000 = 1000w, which is true only if the hundreds digit of 7600t is zero, and that happens only when \( t = 5 \). Sorting this out, it follows that only 538 has an 8 as units digit and a square that ends in 444: 538^2 = 289444. The case where \( n \) has units digit 8 does not lead to any solutions.

19. A six-digit number \( N = abcdef \) has the property that 7(abcdef) = 6(defabc), where both digits \( a \) and \( d \) are non-zero. What is \( N \)?

**Solution:** The number is 461,538. To find it, let \( x \) denote the three-digit number \( abc \) and let \( y \) denote the three-digit number \( def \). Then 7(1000x + y) = 6(1000y + x). This leads to 6994x = 5993b, which is equivalent to 2 \cdot 13 \cdot 269a = 13 \cdot 461b, so we can let see that \( x = 461 \) and \( y = 538 \) work.

20. Let \( a, b, c, d, \) and \( e \) be digits satisfying \( 4 \cdot abcde = 4abcde \). Find all five of the digits.
Solution: Let \( x = abced \). Then \( 4(10x + 4) = 400000 + x \), from which it follows that \( 39x = 399984 \) and \( x = 10256 \).

21. Let \( a, b, c, d, e \) be digits satisfying \( abc \cdot a = bda \) and \( bda \cdot a = cdde \). What is \( cdde \cdot a \)? A reminder about notation: the string of digits \( abc \) can be interpreted two ways, first as \( a \cdot b \cdot c \) and secondly as \( 100a + 10b + c \). To distinguish these two interpretations, we use the underline notation \( abc \) for the latter of these.

Solution: The answer is unique, 2008. This problem is due to Marcin Kuczma. First note that \( abc \cdot a \) is a three-digit number so \( a \leq 3 \) and if \( a = 3 \), the \( b \leq 3 \). If \( a = 1 \), then \( abc \cdot a = 1bc \cdot 1 = abc = bda \) in which case \( bde \cdot a \) is not a four-digit number. Therefore, either \( a = 2 \) or \( a = 3 \). If \( a = 3 \), then \( 3bc \cdot 3 = bda \geq 900 \). This is impossible because \( abc \cdot a \) is a three-digit number. Thus, \( a = 2 \).

Now we can write \( 2bc \cdot 2 = bd2 \), and in decimal(place value) form \( (200 + 10b + c) \cdot 2 = 100b + 10d + 2 \). This simplifies to \( 80b - c + 10d = 398 \). Thus \( 39 \leq 8b + d \leq 40 \) and it follows from this that \( 30 \leq 8b \leq 40 \) and that \( 3.75 \leq b \leq 5 \). In other words, either \( b = 4 \) or \( b = 5 \). Its easy to see that \( b = 4 \) implies that \( d = 8 \) and \( c = 2 \), which means that \( abc \cdot a = 484 = bda \) cannot be satisfied. Trying \( b = 5 \), we see that \( 25c \cdot 2 = 5d2 \) which translates to \( (250 + c) \cdot 2 = 502 + 10d \). From which it follows that \( c = 1 \) and \( d = 0 \). In this case \( abc = 251 \), \( abc \cdot a = bda = 502 \), and \( bda \cdot a = 502 \cdot 2 = 1004 = cdde \) and so \( e = 4 \). Finally \( cdde \cdot a = 1004 \cdot 2 = 2008 \).

22. Let \( N = abcdef \) be a six-digit number such that \( defabc \) is six times the value of \( abcdef \). What is the sum of the digits of \( N \)?

Solution: 27. Let \( x = abc \) and \( y = def \). Then \( 100y + x = 6(1000x + y) \), from which it follows that \( 857x = 142y \). Since \( 857 \) and \( 142 \) are relatively prime, \( 857 \) must divide \( y \). Since \( y \) is a three-digit number, \( y \) must be \( 857 \). And \( x = 142 \). The sum of the digits is \( 8 + 5 + 7 + 1 + 4 + 2 = 27 \).

23. Maximizing Products

(a) Using all nonzero digits each once, build two numbers \( A \) and \( B \) so that \( A \cdot B \) is as large as possible.

Solution: There are two principles at work here. First, among all products of two numbers whose sum is fixed, the largest is that
for which the numbers are as close to each other as possible. For example, the maximum value of $xy$ over all pairs $x, y$ such that $x + y = S$ is $(S/2)^2$. The other principle is place value. In the case of two numbers, the best we can do it $9642$ and $87531$, so $A \cdot B = 843973902$

(b) Using all nonzero digits each once, build three numbers $A, B$ and $C$ so that $A \cdot B \cdot C$ is as large as possible.

**Solution:** Here, the best is $941 \cdot 852 \cdot 763 = 611721516$.

(c) Using all nonzero digits each once, build four numbers $A, B, C$ and $D$ so that $A \cdot B \cdot C \cdot D$ is as large as possible.

**Solution:** The best we can do here is $92 \cdot 83 \cdot 74 \cdot 651 = 367856664$. You can prove that this is best by first solving the problem in the case of only 8 available digits. Then we get $92 \cdot 83 \cdot 74 \cdot 65$. Next suppose digit 0 was available. Clearly it doesn’t matter where it goes. Now if 1 is available, argue that you can do better putting it with the 65 than elsewhere.

(d) If we build five two-digit numbers using each of the digits 0 through 9 exactly once, and the product of the five numbers is maximized, find the greatest number among them.

**Solution:** 90. As we saw above, there are two principles at work here. First we want to use each large digit where it counts the most, that is, in the tens place. Second, when the sum of two (or more) numbers is fixed, we can make the product as large as possible by making the numbers as close together as possible. So we must take the digits 5 through 9 as tens digits and 0 through 4 as units digits. To make the numbers as close as possible, use 0 with 9, 1 with 8, etc. to get $90 \cdot 81 \cdot 72 \cdot 63 \cdot 54$. 
24. Calling All Digits

(a) Using each nonzero digit exactly once, create three 3-digit numbers $A$, $B$, and $C$, such that $A + B = C$.

**Solution:** Let $S(N)$ denote the sum of the digits of $N$. We can use modulo nine arithmetic to cut down on the possibilities. Note that $A + B + C \equiv 1 + 2 + 3 + \cdots + 9 = 45 \equiv 0 \pmod{9}$. Also, $A + B \equiv C \pmod{9}$. Adding the two congruences, $A + B + C \equiv 0 \pmod{9}$ and $A + B - C \equiv 0 \pmod{9}$ yields $2(A + B) \equiv 0 \pmod{9}$, since congruences can be added and subtracted. It follows that $C \equiv 0 \pmod{9}$. So the sum of the digits of $C$ is either 9 or 18 (27 is impossible). There are 18 numbers satisfying $S(C) = 9$. They are permutations (of the digits) of the numbers 126, 135, and 234. These 18 numbers are all less than the smallest sum obtainable by arranging the other 6 nonzero digits. For example, 621 is less that 357 + 489.

There are seven three-element subsets of $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ that have a sum of member equal to 18: $\{1, 8, 9\}$, $\{2, 7, 9\}$, $\{3, 6, 9\}$, $\{3, 7, 8\}$, $\{4, 5, 9\}$, $\{4, 6, 8\}$, $\{5, 6, 7\}$. Each of these give rise to six candidates for $C$. The 8 numbers less than 400 are too small to be $C$, but it appears that all the other 34 numbers work. For example $237 + 654 = 891$, $327 + 654 = 981$, $276 + 543 = 819$, $275 + 643 = 918$, $586 + 143 = 729$, etc. Of course, once you find one set of three numbers that work, you can quickly find seven other by interchanging pairs of digits.

(b) Again using each nonzero digit exactly once, create three 3-digit numbers $A$, $B$, and $C$ that are in the ratio 1 : 3 : 5.

**Solution:** I don’t have this problem completely solved, but I know a lot about it. Here’s what I know. For $A$, $B$, and $C$ to be in the ratio 1 : 3 : 5, we must have $B = 3A$ and $C = 5A$. Since $C < 1000$, it follows that $A < 1000/5 = 200$, so the hundreds digit of $A$ is 1. Since $C$ is a multiple of 5, and the digit 0 is not available, the units digit of $C$ must be 5 and the units digit of $A$ is odd. One solution is $A = 129$, $B = 387$, and $C = 645$. My friend Arthur Holshouser reports that this solution is unique.

(c) Again using each nonzero digit exactly once, create three 3-digit numbers $A$, $B$, and $C$ that are in the ratio 1 : 2 : 3.
Solution: There are four solutions and the have interesting symmetries: \{192, 384, 576\}, \{219, 438, 657\}, \{273, 546, 819\}, and \{327, 654, 981\}.

(d) Again using each nonzero digit exactly once, create three 3-digit numbers \(A, B,\) and \(C\) that are in the ratio 4 : 5 : 6.

Solution: The solution is unique: \{492, 615, 738\}. You can find it this way. Let \(A = abc, B = def, C = ghi.\) Since \(a + b + c + \cdots + i = 45\) is a multiple of 9, it follows that \(A + B + C\) is also a multiple of 9. Let \(x = (A + B + C) / 15.\) Then \(A = 4x, B = 5x\) and \(C = 6x.\) Now \(B\) has the form \(de5\) because \(B\) is a multiple of 5 that cannot have a digit 0. Note that \(A + B = 9x\) so \(A + B\) is a multiple of 9. That means \(C\) must also be a multiple of 9, which means \(x\) must be a multiple of 3. This means that \(d + e + 5\) must be one of the numbers 6, 9, 12, 15, or 18. All these possibilities are easily eliminated except \(d + e = 7\) and \(d + e = 10.\) However, with some heavy-duty paper and pencil arithmetic, we can eliminate \(d + e = 10\) and eventually find \(d = 6, e = 1\) from which it follows that \(x = 123, A = 492, B = 615,\) and \(C = 738.\)

(e) Again using each nonzero digit exactly once, create three 3-digit numbers \(A, B,\) and \(C\) that are in the ratio 3 : 7 : 8.

Solution: There are just two solutions and each uses the same digit sets: \{213, 497, 568\}, \{321, 749, 856\}.

(f) Are there any more single digit ratios \(a : b : c\) for which the nine nonzero digits can be used to build three numbers \(A, B,\) and \(C\) in the ratio \(a : b : c.\)

Solution: Arthur Holshouser reports that the ratios 1 : 2 : 3, 1 : 3 : 5, 4 : 5 : 6, and 3 : 7 : 8 are the only ones that can be satisfied. I do not have a proof of this.

(g) Using the ten digits each exactly once, create three numbers \(A, B,\) and \(C,\) such that \(A + B = C.\)

Solution: One solution is \(A = 734, B = 895,\) and \(C = 1602.\) Lots of others can be found from this one. Must \(C\) be a multiple of 9? Look at \(A + B \mod 9,\) and \(A + B + C \mod 9.\)

(h) Use each of the nine nonzero digits exactly once to construct three prime numbers \(A, B,\) and \(C\) such that the sum \(A + B + C\) is as small as possible.
Place Value Problems

Solution: $149 + 587 + 263 = 999$. Give an argument that you can do no better than this.

(i) Now use each of the ten digits exactly once to construct three prime numbers $A, B$ and $C$ such that the sum $A + B + C$ is as small as possible.

Solution: $1069 + 457 + 283 = 1809$. Give an argument that you can do no better than this.

25. The number $N = 123456789101112 \ldots 999$ is formed stringing together all the numbers from 1 to 999. What is the product of the $2007^\text{th}$ and $2008^\text{th}$ digits of $N$?

Solution: The $2007^\text{th}$ digit is 5 and the $2008^\text{th}$ digit is 7, so the product is 35.

26. The number $N = 37! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 37$ is a 44-digit number. The first 33 digits are $K = 137637530912263450463159795815809$. In fact, $N = K \cdot 10^{11} + L \cdot 10^8$, where $L$ is less than 1000. Find the number $L$.

Solution: $37! = 1376375309122634504631597958158092000000$. So the number $L$ is 24. Let $L = abc$. There are three solutions. One depends on the fact that $1001 = 7 \cdot 11 \cdot 13$. Write $37!$ in base 1000, so that each digit is of the form $uvw$, where $u, v,$ and $w$ are decimal digits. Thus $37! = 13 \cdot (1000)^{14} + 763 \cdot (1000)^{13} + \ldots + 9ab \cdot (1000)^3 + c00 \cdot 1000^2$.

But since $1000^n = (1001 - 1)^n = 1001^n - n \cdot 1001^{n-1} + \ldots + (-1)^n$, we can see that only the last term fails to be a multiple of 1001. This means that $uvw \cdot 1000^n \equiv (-1)^n \pmod{1001}$. Thus $37! \equiv 13 + 753 + 226 + 46 + 979 + 580 + 10c - (763 + 91 + 345 + 315 + 581 + 900 + 10a + b) \equiv -398 + 100c - 10a - b \equiv 0 \pmod{1001}$. This can happen only when $c = 4, a = 0$ and $b = 2$, so $L = 24$. The second method depends upon the fact that 999 = 27 · 37, so 37! is a multiple of 999. Once again, express the number 37! in base 1000, where $L = abc$. Since 1000 $\equiv 1 \pmod{999}$, the number 37! is the sum of its digits, mod 999. Hence, $37! \equiv 13 + 763 + 753 + 901 + 226 + 345 + 46 + 315 + 979 + 581 + 580 + 9ab + c00 + 000 + 000 \equiv 5592 + 9ab + c00 \equiv 0 \pmod{999}$. The only way this can happen is $5592 + 9ab + c00 = 6 \cdot 999 = 5994$, from which it follows that $ab + c00 = 402$, which means $c = 4, a = 0$ and $b = 2$. The third and easiest method to understand uses brute force to get the rightmost digit, and then mod 9 and mod 11 arithmetic to find the other two digits.
27. Find the smallest integer multiple of 84 and whose decimal representation uses just the two digits 6 and 7.

Solution: Let $N$ denote the number. The rightmost two digits must be 76, so the number is $100k + 76$ for some integer $k$. Since 3 divides 84, $N$ must be a multiple of 3, so the sum of the digits of $N$ must be a multiple of 3. It follows that the number of 7’s in $N$ must be a multiple of 3. Let’s try for three 7’s. Does 7776 work? No, it’s not a multiple of 7. Next we need to try three 7’s and two 6’s. There are three numbers to try, 77676, 76776, and 67776. Notice that $77777 - 76776 = 1001 = 7 \cdot 11 \cdot 13$, so 76776 is a multiple of 7, 3 and 4. Since these three numbers are relatively prime, 76776 is a multiple of 84.

28. (Mathcounts 2009) Find a six-digit number $abcdef$ such that $4 \cdot abcdef = 3 \cdot defabc$.

Solution: Note that $\cdot abcdef = 4000 \cdot abc + 4 \cdot def$ while $3 \cdot defabc = 3000 \cdot def + 3 \cdot abc$. If we let $x = abc$ and $y = def$, then $4000x + 4y = 3000y + 3x$, which is equivalent to $3997x = 2996y$. Both the numbers 3997 and 2996 are multiples of 7, so $571x = 428y$. Therefore we can simply let $y = 571$ and $x = 428$ to get the six-digit number 428571.

29. Find the greatest 9-digit number whose digits’ product is 9!.

Solution: We use the Greedy Algorithm to construct the largest possible integer of this form. Proceeding place by place, starting at the left, we select the largest available digit. Since $9! = 27 \cdot 34 \cdot 5 \cdot 7$, we can begin with a maximum of two nines followed by two eights, leaving 2 \cdot 5 \cdot 7 in the product. We then can make one seven, no sixes, one five, no fours or threes, and one two. To create a nine digit number, we add two trailing ones. Thus, our constructed number is 998, 875, 211. Solution by Jonathan Schneider.

30. Find values for each of the digits $A, B$ and $C$.

\[
\begin{align*}
BA \\
AB \\
+ AB \\
CAA
\end{align*}
\]
**Solution:** Clearly $C = 1$ since otherwise $2A + B \geq 20$ in which case $A + 2B$ cannot have units digit $A$. To make $A + 2B$ have units digit $A$, we must have $B = 5$. Then $50 + A + 20A + 10 = 100 + 11A$. From this it follows that $A = 4$. 