AN EXACT SOLUTION TO A PROBLEM OF AXISYMMETRIC TORSION OF AN ELASTIC SPACE WITH A SPHERICAL CRACK

By

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Abstract. It is a well-known fact that the three-dimensional axisymmetric dilatation problem for an elastic space containing a spherical crack has an explicit solution, found and analyzed in [1]–[3]. The corresponding torsion problem, however, also of interest, has not apparently been investigated. To remedy this shortage we present here an exact solution to this problem, which incidentally reveals a remarkable effect concerning the stress intensity.

Denoting by \((R, \theta, \phi)\) a spherical coordinate system, let a crack be situated at \(R = a, 0 \leq \theta \leq \theta_0, 0 \leq \phi < 2\pi\), in an elastic medium twisted at infinity. Then the displacement vector is \(\{0, 0, v_{\phi} = v(R, \theta)\}\), and the only nonvanishing stress components are

\[
\tau_{R\phi} = GR \frac{\partial}{\partial R} \left( \frac{v}{R} \right), \quad \tau_{\theta\phi} = \frac{G}{R} \left( \frac{\partial v}{\partial \theta} - v \cot \theta \right),
\]

where \(G\) is the shear modulus. We look for a solution in the form

\[
v(R, \theta) = u^*(R, \theta) + u(R, \theta),
\]

where \(u^* = \alpha R^2 \sin \theta \cos \theta\) is the displacement in the crack-free medium, and \(\alpha\) is the twist angle per unit length. Setting

\[
u = \begin{cases} u_i(R, \theta), & 0 \leq R < a, \\ u_e(R, \theta), & R > a \end{cases}
\]

we arrive at the following mixed boundary-value problem:

\[
\nabla^2 u - \frac{u}{R^2 \sin^2 \theta} = 0, \quad R \neq a;
\]

\[
u_e \to 0 \quad \text{as } R \to \infty;
\]

\[
\tau_{R\phi}^{(i)}(a, \theta) = \tau_{R\phi}^{(e)}(a, \theta) = -\alpha a G \sin \theta \cos \theta, \quad 0 \leq \theta < \theta_0;
\]

\[
\tau_{R\phi}^{(i)}(a, \theta) = \tau_{R\phi}^{(e)}(a, \theta), \quad \theta_0 < \theta \leq \pi;
\]

\[
u_i(a, \theta) = \nu_e(a, \theta), \quad \theta_0 < \theta \leq \pi.
\]
A solution that satisfies Eq. (4) and is compatible with the condition (5) at infinity reads
\[
   u_i = \alpha a^2 \sum_{n=1}^{\infty} A_n \left( \frac{R}{a} \right)^n P_n^1(\cos \theta), \quad 0 \leq R < a; \quad (9)
\]
\[
   u_e = \alpha a^2 \sum_{n=1}^{\infty} B_n \left( \frac{a}{R} \right)^{n+1} P_n^1(\cos \theta), \quad R > a. \quad (10)
\]

Here \( P_n^1(\cos \theta) \) is the associated Legendre function [4] and \( A_n, B_n \) are the unknown dimensionless coefficients. The stress continuity conditions (6)–(7) relate \( B_n \) to \( A_n \) by
\[
   B_n = -\frac{n-1}{n+2} A_n, \quad n = 1, 2, \ldots, \quad (11)
\]
and then (6) and (8) yield the following dual series associated with Legendre functions:
\[
   \sum_{n=1}^{\infty} (n-1) A_n P_n^1(\cos \theta) = -\sin \theta \cos \theta, \quad 0 \leq \theta < \theta_0; \quad (12)
\]
\[
   \sum_{n=1}^{\infty} \frac{2n+1}{n+2} A_n P_n^1(\cos \theta) = 0, \quad \theta_0 \leq \theta < \pi. \quad (13)
\]

This system of dual series differs from the one investigated in [5] and therefore, according to [3], we represent the solution of (12)–(13) in the form
\[
   A_n = \frac{n+2}{2n(n+1)} \int_{0}^{\theta_0} \psi(t) \cos \left( n + \frac{1}{2} \right) t \, dt \quad (14)
\]
with \( \psi(t) \in C_{[0, \theta_0]} \).

Substituting (14) back in (12)–(13) and resorting to Fourier expansion in Legendre polynomials [4] we find that (13) holds if \( \psi(t) \) satisfies the condition
\[
   \int_{0}^{\theta_0} \psi(t) \cos \frac{t}{2} \, dt = 0. \quad (15)
\]

After integration of (12) with respect to \( \theta \), and taking into account Mehler's integral representation [4] with the necessary discontinuous sum for Legendre polynomials [4], we obtain the Abel integral equation
\[
   \int_{0}^{\theta} \left[ \psi(x) - \frac{4}{\pi} \int_{0}^{\theta_0} K(x, t) \psi(t) \right] \frac{dt \, dx}{\sqrt{2} \cos x - 2 \cos \theta} = C - \sin^2 \theta, \quad 0 \leq \theta < \theta_0, \quad (16)
\]
where
\[
   K(x, t) = \sum_{n=1}^{\infty} \frac{\cos(n + \frac{1}{2})x \cos(n + \frac{1}{2})t}{n(n+1)}
\]
\[
   = 2 \cos \frac{t}{2} \left[ (\pi - x) \sin \frac{x}{2} - \cos \frac{x}{2} \right] - 2t \sin \frac{t}{2} \cos \frac{x}{2} \quad (17)
\]
and \( C \) is an unknown constant.
Solution of Eq. (16) leads to a Fredholm integral equation of the second kind:

$$
\psi(x) - \frac{4}{\pi} \int_0^{\theta_0} K(x, t) \psi(t) \, dt = \frac{4}{3 \pi} \cos \frac{5x}{2} + D \cos \frac{x}{2}, \quad 0 \leq x < \theta,
$$

(18)

where \( D \) is another unknown constant. The kernel (17) of this equation is a degenerate one and thus permits an explicit solution to Eq. (18) that satisfies condition (15) and specifies the value of the unknown constant \( D \), namely

$$
\psi(x) = \frac{4}{3 \pi} \left[ \cos \frac{5x}{2} - \frac{2 \sin 3\theta_0 + 3 \sin 2\theta_0}{6(\theta_0 + \sin \theta_0)} \cos \frac{x}{2} \right].
$$

(19)

The shear stress \( \tau_{R\phi} \) on the surface \( R = a \) follows as

$$
\tau_{R\phi}(a, \theta) = \alpha a G \left[ \sin \theta \cos \theta - \frac{1}{2} \sin \theta \int_0^{\theta_0} \frac{\psi(t) \, dt}{(2 \cos t - 2 \cos \theta)^{3/2}} \right. \\
\left. + \frac{1}{\sin \theta} \int_0^{\theta_0} \psi(t)(2 \cos t - 2 \cos \theta)^{1/2} \, dt \right], \quad \theta_0 < \theta \leq \pi,
$$

(20)

and the integrals here may be calculated explicitly with the aid of (19).

This formula results in an expression for the stress intensity factor \( K_{III} \) in the form

$$
K_{III} = \lim_{\theta \to \theta_0} \tau_{R\phi} \sqrt{2\pi a(\theta - \theta_0)}
$$

$$
= \frac{2\alpha a^{3/2} G}{3\sqrt{\pi} \sin^{1/2} \theta_0} \left[ \frac{2 \sin 3\theta_0 + 3 \sin 2\theta_0}{6(\theta_0 + \sin \theta_0)} \cos \frac{\theta_0}{2} - \cos \frac{5\theta_0}{2} \right].
$$

(21)

Figure 1 (see p. 682) illustrates the dependence of \( K_{III} = K_{III}(\alpha a^{3/2} G)^{-1} \) on the crack angle \( \theta_0 \), which appears to be nonmonotonic. At \( \theta_0 \approx 106^\circ \), \( K_{III} \) vanishes completely and then changes its sign. Its maximum and minimum values are at \( \theta_0 \approx 61^\circ \) and \( 152^\circ \), respectively; the highest value of \( |K_{III}| \) is at \( \theta_0 = 152^\circ \).

Finally, it is worthwhile to consider the limiting case \( \theta_0 \to 0, a \to \infty \) such that \( a\theta_0 \to b \) (transition to a penny-shaped crack). In that case, we find from (21)

$$
K_{III} = \frac{4}{3\sqrt{\pi}} \alpha b^{3/2} G,
$$

(22)

in agreement with the known result from (6).
Fig. 1. Dependence of the normalized stress intensity factor $\bar{K}_{III}$ on the crack angle.

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REFERENCES


CORRIGENDA: An exact solution to a problem of axisymmetric torsion of an elastic space with a spherical crack

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The sum (17) in the paper was calculated incorrectly. Its value should be

\[
K(x, t) = \sum_{n=1}^{\infty} \frac{\cos \left(n + \frac{1}{2}\right) x \cos \left(n + \frac{1}{2}\right) t}{n(n+1)} = \cos \frac{x}{2} \cos \frac{t}{2} + x \sin \frac{x}{2} \cos \frac{t}{2} + t \sin \frac{t}{2} \cos \frac{x}{2}
\]

\[
- \begin{cases} 
\pi \sin \frac{x}{2} \cos \frac{t}{2}, & 0 \leq t \leq x \leq \theta_0, \\
\pi \sin \frac{t}{2} \cos \frac{x}{2}, & 0 \leq x \leq t \leq \theta_0.
\end{cases}
\]

Consequently, the function \( \psi(x) \) becomes

\[
\psi(x) = \frac{2}{\pi} \cos \frac{5x}{2} - \frac{2}{3\pi} \left(4 \cos \theta_0 - 1\right) \cos \frac{3x}{2},
\]

and the stress intensity factor (21) reads

\[
K_{\text{III}} = \frac{\alpha a^{3/2} G}{\sqrt{\pi} \sin \theta_0} \left(\frac{4 \cos \theta_0 - 1}{3} \cos \frac{3\theta_0}{2} - \cos \frac{5\theta_0}{2}\right).
\]

The dependence of \( \tilde{K}_{\text{III}} = K_{\text{III}}(\alpha a^{3/2} G)^{-1} \) on the crack angle \( \theta_0 \) is given below. Its maximum value is attained at the angle \( 2 \arccos \sqrt{10} \approx 75.5^\circ \).

![Graph showing the dependence of \( \tilde{K}_{\text{III}} \) on \( \theta_0 \).]

**Fig.1.** Dependence of the normalized stress intensity factor \( \tilde{K}_{\text{III}} \) on the crack angle \( \theta_0 \).