Dividing Polynomials
Long Division of Polynomials

• **Arrange the terms** of both the dividend and the divisor in descending powers of any variable.
• **Divide** the first term in the dividend by the first term in the divisor. The result is the first term of the quotient.
• **Multiply** every term in the divisor by the first term in the quotient. Write the resulting product beneath the dividend with like terms lined up.
• **Subtract** the product from the dividend.
• **Bring down** the next term in the original dividend and write it next to the remainder to form a new dividend.
• Use this new expression as the dividend and repeat this process until the remainder can no longer be divided. This will occur when the degree of the remainder (the highest exponent on a variable in the remainder) is less than the degree of the divisor.
**Text Example**

**Divide** \(4 - 5x - x^2 + 6x^3\) **by** \(3x - 2\).

**Solution** We begin by writing the divisor and dividend in descending powers of \(x\). Next, we consider how many times \(3x\) divides into \(6x^3\).

Multiply.  
\[
\begin{array}{c}
3x - 2 \quad 6x^3 - x^2 - 5x + 4 \\
\Theta 6x^3 \oplus 4x^2 \\
3x^2 - 5x
\end{array}
\]

Divide: \(6x^3/3x = 2x^2\).

Multiply: \(2x^2(3x - 2) = 6x^3 - 4x^2\).

Subtract \(6x^3 - 4x^2\) from \(6x^3 - x^2\) and bring down \(-5x\).

Now we divide \(3x^2\) by \(3x\) to obtain \(x\), multiply \(x\) and the divisor, and subtract.

Multiply.  
\[
\begin{array}{c}
3x - 2 \quad 6x^3 - x^2 - 5x + 4 \\
\Theta 6x^3 - 4x^2 \\
3x^2 - 5x
\end{array}
\]

Divide: \(3x^2/3x = x\).

Multiply: \(x(3x - 2) = 3x^2 - 2x\).

Subtract \(3x^2 - 2x\) from \(3x^2 - 5x\) and bring down \(4\).
Text Example cont.

Divide \(4 - 5x - x^2 + 6x^3\) by \(3x - 2\).

**Solution**  Now we divide \(-3x\) by \(3x\) to obtain \(-1\), multiply \(-1\) and the divisor, and subtract.

Multiply: \(-1(3x - 2) = -3x + 2\).

Subtract \(-3x + 2\) from \(-3x + 4\), leaving a remainder of 2.
The Division Algorithm

If \( f(x) \) and \( d(x) \) are polynomials, with \( d(x) = 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), then there exist unique polynomials \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x) \cdot q(x) + r(x).
\]

The remainder, \( r(x) \), equals 0 or its is of degree less than the degree of \( d(x) \). If \( r(x) = 0 \), we say that \( d(x) \) divides evenly into \( f(x) \) and that \( d(x) \) and \( q(x) \) are factors of \( f(x) \).
Example

• Divide:

\[
\begin{array}{c}
3x^3 - 2x^2 + 4x - 3 \\
\hline
x^2 + 3x + 3
\end{array}
\]
Example cont.

\[ x^2 + 3x + 3 \mid 3x^3 - 2x^2 + 4x - 3 \]
Example cont.

\[ 3x^2 + 3x + 3 \mid \overline{3x^3 - 2x^2 + 4x - 3} \]
Example cont.

\[ \begin{array} \{ x^2 + 3x + 3 \} \end{array} \]

\[
\begin{array} \{ 3x \} \\
3x^3 - 2x^2 + 4x - 3 \\
3x^3 + 9x^2 + 9x
\end{array}
\]
Example cont.

\[ x^2 + 3x + 3 \bigg| \begin{array}{c}
3x^3 - 2x^2 + 4x - 3 \\
3x^3 + 9x^2 + 9x
\end{array} \\
-11x^2 - 5x - 3 \]
Example cont.

\[
x^2 + 3x + 3 \overline{3x^3 - 2x^2 + 4x - 3}
\]

\[
-((3x^3 + 9x^2 + 9x))
\]

\[-11x^2 - 5x - 3\]
Example cont.

$$x^2 + 3x + 3 \div \overline{3x^3 - 2x^2 + 4x - 3}$$

$$3x^3 + 9x^2 + 9x$$

$$-11x^2 - 5x - 3$$

$$-11x^2 - 33x - 33$$
Example cont.

\[ x^2 + 3x + 3 \div 3x^3 - 2x^2 + 4x - 3 \]

\[
\begin{array}{c}
3x^3 + 9x^2 + 9x \\
-11x^2 - 5x - 3 \\
-11x^2 - 33x - 33 \\
\hline
28x + 30
\end{array}
\]
Example cont.

\[
\frac{3x^3 - 2x^2 + 4x - 3}{x^2 + 3x + 3} =
\]

\[
(3x - 11) + \frac{28x + 30}{x^2 + 3x + 3}
\]
Synthetic Division

To divide a polynomial by \( x - c \)

1. Arrange polynomials in descending powers, with a 0 coefficient for any missing terms.

\[ x - 3 \] \( x^3 + 4x^2 - 5x + 5 \]

2. Write \( c \) for the divisor, \( x - c \). To the right, write the coefficients of the dividend.

\[ \begin{array}{cccc}
3 & 1 & 4 & -5 & 5 \\
\end{array} \]

3. Write the leading coefficient of the dividend on the bottom row.

\[ \begin{array}{cccc}
3 & 1 & 4 & -5 & 5 \\
\end{array} \]

Bring down 1.

\[ 1 \]

4. Multiply \( c \) (in this case, 3) times the value just written on the bottom row. Write the product in the next column in the 2nd row.

\[ \begin{array}{cccc}
3 & 1 & 4 & -5 & 5 \\
1 & 3 & & \end{array} \]

Multiply by 3.
5. Add the values in this new column, writing the sum in the bottom row.

6. Repeat this series of multiplications and additions until all columns are filled in.

7. Use the numbers in the last row to write the quotient and remainder in fractional form. The degree of the first term of the quotient is one less than the degree of the first term of the dividend. The final value in the row is the remainder.

\[ 1x^2 + 7x + 16 + \frac{53}{x - 3} \]
Use synthetic division to divide $5x^3 + 6x + 8$ by $x + 2$.

**Solution** The divisor must be in the form $x - c$. Thus, we write $x + 2$ as $x - (-2)$. This means that $c = -2$. Writing a 0 coefficient for the missing $x^2$-term in the dividend, we can express the division as follows:

$$
\begin{array}{c}
5x^3 + 0x^2 + 6x + 8 \\
\end{array} 
\overline{\begin{array}{c}
x - (-2)
\end{array}}
$$

Now we are ready to set up the problem so that we can use synthetic division. This is $c$ in $x - (-2)$. Use the coefficients of the dividend in descending powers of $x$. 


Solution  We begin the synthetic division process by bringing down 5. This is followed by a series of multiplications and additions.

1. Bring down 5.

2. Multiply: 
   \[-2(5) = -10.\]
   Add: 
   \[0 + (-10) = -10.\]

3. Multiply: 
   \[-2(26) = -52.\]
   Add: 
   \[6 + 26 = 32.\]

4. Multiply: 
   \[-2(-10) = 20.\]
   Add: 
   \[8 + (-52) = -44.\]
The numbers in the last row represent the coefficients of the quotient and the remainder. The degree of the first term of the quotient is one less than that of the dividend. Because the degree of the dividend is 3, the degree of the quotient is 2. This means that the 5 in the last row represents $5x^2$.

Thus,

\[
x + 2 \overline{\underline{5x^3 + 0x^2 + 6x + 8}} \quad \underline{44}
\]

\[
x + 2 \overline{\underline{5x^2 - 10x + 26}} \quad \underline{44}
\]
The Remainder Theorem

• If the polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$. 
The Factor Theorem

• Let $f(x)$ be a polynomial.
• If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
• If $x - c$ is a factor of $f(x)$, then $f(c) = 0$. 
Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ given that 3 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

**Solution** We are given that $f(3) = 0$. The Factor Theorem tells us that $x - 3$ is a factor of $f(x)$. We’ll use synthetic division to divide $f(x)$ by $x - 3$.

Equivalently, $2x^3 - 3x^2 - 11x + 6 = (x - 3)(2x^2 + 3x - 2)$
Solution
Now we can solve the polynomial equation.

\[2x^3 - 3x^2 - 11x + 6 = 0\]
This is the given equation.

\[(x - 3)(2x^2 + 3x - 2) = 0\]
Factor using the result from the synthetic division.

\[(x - 3)(2x - 1)(x + 2) = 0\]
Factor the trinomial.

\[x - 3 = 0 \quad \text{or} \quad 2x - 1 = 0 \quad \text{or} \quad x + 2 = 0\]
Set each factor equal to 0.

\[x = 3 \quad x = \frac{1}{2} \quad x = -2\]
Solve for x.

The solution set is \{-2, \frac{1}{2}, 3\}.
Dividing Polynomials