Rational Functions and Their Graphs
Example

• Find the Domain of this Function.
• Solution:
• The domain of this function is the set of all real numbers not equal to 3.

\[ f(x) = \frac{x + 7}{x - 3} \]
# Arrow Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \rightarrow a^+$</td>
<td>$x$ approaches $a$ from the right.</td>
</tr>
<tr>
<td>$x \rightarrow a^-$</td>
<td>$x$ approaches $a$ from the left.</td>
</tr>
<tr>
<td>$x \rightarrow \infty$</td>
<td>$x$ approaches infinity; that is, $x$ increases without bound.</td>
</tr>
<tr>
<td>$x \rightarrow -\infty$</td>
<td>$x$ approaches negative infinity; that is, $x$ decreases without bound.</td>
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</table>
Definition of a Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function $f$ if $f(x)$ increases or decreases without bound as $x$ approaches $a$.

\[ f(x) \to \infty \text{ as } x \to a^+ \quad f(x) \to \infty \text{ as } x \to a^- \]

Thus, $f(x) \to \infty$ or $f(x) \to -\infty$ as $x$ approaches $a$ from either the left or the right.
Definition of a Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function $f$ if $f(x)$ increases or decreases without bound as $x$ approaches $a$. Thus, $f(x) \to -\infty$ as $x \to a^+$ and $f(x) \to -\infty$ as $x \to a^-$. Therefore, $f(x) \to -\infty$ or $f(x) \to -\infty$ as $x$ approaches $a$ from either the left or the right.
Locating Vertical Asymptotes

- If \( f(x) = \frac{p(x)}{q(x)} \) is a rational function in which \( p(x) \) and \( q(x) \) have no common factors and \( a \) is a zero of \( q(x) \), the denominator, then \( x = a \) is a vertical asymptote of the graph of \( f \).
The line $y = b$ is a horizontal asymptote of the graph of a function $f$ if $f(x)$ approaches $b$ as $x$ increases or decreases without bound.
Locating Horizontal Asymptotes

Let \( f \) be the rational function given by

\[
f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0
\]

The degree of the numerator is \( n \). The degree of the denominator is \( m \).

5. If \( n < m \), the x-axis, or \( y = 0 \), is the horizontal asymptote of the graph of \( f \).

6. If \( n = m \), the line \( y = \frac{a_n}{b_m} \) is the horizontal asymptote of the graph of \( f \).

7. If \( n > m \), the graph of \( f \) has no horizontal asymptote.
Strategy for Graphing a Rational Function

• Suppose that \( f(x) = \frac{p(x)}{q(x)} \),

where \( p(x) \) and \( q(x) \) are polynomial functions with no common factors.

1. Determine whether the graph of \( f \) has symmetry.
   \[ f(-x) = f(x) : y\text{-axis symmetry} \]
   \[ f(-x) = -f(x) : \text{origin symmetry} \]

5. Find the \( y \)-intercept (if there is one) by evaluating \( f(0) \).

6. Find the \( x \)-intercepts (if there are any) by solving the equation \( p(x) = 0 \).

7. Find any vertical asymptote(s) by solving the equation \( q(x) = 0 \).

8. Find the horizontal asymptote (if there is one) using the rule for determining the horizontal asymptote of a rational function.

9. Plot at least one point between and beyond each \( x \)-intercept and vertical asymptote.

10. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.
Sketch the graph of

\[ f(x) = \frac{2x - 3}{5x + 10} \]
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- the graph has no symmetry
\[ f(x) = \frac{2x - 3}{5x + 10} \]

- the graph has no symmetry
- The y-intercept is (0, -3/10)
\[
f(x) = \frac{2x - 3}{5x + 10}
\]

- the graph has no symmetry
- The y-intercept is \((0, -3/10)\)
- The x-intercept is \((3/2, 0)\)
The graph has no symmetry.

The y-intercept is \((0, -\frac{3}{10})\).

The x-intercept is \((\frac{3}{2}, 0)\).

The vertical asymptote is \(x = -2\).

\[ f(x) = \frac{2x - 3}{5x + 10} \]
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- the graph has no symmetry
- The y-intercept is (0, -3/10)
- The x-intercept is (3/2, 0)
- The vertical asymptote is \( x = -2 \)
- The horizontal asymptote is \( y = \frac{2}{5} \)
\[ f(x) = \frac{2x - 3}{5x + 10} \]

- the graph has no symmetry
- The y-intercept is \((0, -3/10)\)
- The x-intercept is \((3/2, 0)\)
- The vertical asymptote is \(x = -2\)
- The horizontal asymptote is \(y = 2/5\)
- Test points include \((-3, 9/5), (0, -3/10), (2, 1/20)\)
\[ f(x) = \frac{2x - 3}{5x + 10} \]
Text Example

Find the slant asymptote of \( f(x) = \frac{x^2 - 4x - 5}{x - 3} \)

Solution  Because the degree of the numerator, 2, is exactly one more than the degree of the denominator, 1, the graph of \( f \) has a slant asymptote. To find the equation of the slant asymptote, divide \( x - 3 \) into \( x^2 - 4x - 5 \).

The equation of the slant asymptote is \( y = x - 1 \). Using our strategy for graphing rational functions, the graph is shown.
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