Exponential Functions
Definition of the Exponential Function

The exponential function $f$ with base $b$ is defined by

$$f(x) = b^x \quad \text{or} \quad y = b^x$$

Where $b$ is a positive constant other than $1$ and $x$ is any real number.

Here are some examples of exponential functions.

- $f(x) = 2^x$ Base is 2.
- $g(x) = 10^x$ Base is 10.
- $h(x) = 3^{x+1}$ Base is 3.
The exponential function \( f(x) = 13.49(0.967)^x - 1 \) describes the number of O-rings expected to fail, \( f(x) \), when the temperature is \( x \)°F. On the morning the Challenger was launched, the temperature was 31°F, colder than any previous experience.

Find the number of O-rings expected to fail at this temperature.

**Solution**  
Because the temperature was 31°F, substitute 31 for \( x \) and evaluate the function at 31.

\[
\begin{align*}
  f(x) &= 13.49(0.967)^x - 1 & \text{This is the given function.} \\
  f(31) &= 13.49(0.967)^{31} - 1 & \text{Substitute 31 for } x.
\end{align*}
\]

Press \( .967^\text{31} \) on a graphing calculator to get \( .353362693426 \). Multiply this by 13.49 and subtract 1 to obtain

\[
f(31) = 13.49(0.967)^{31} - 1 = 3.77
\]
Characteristics of Exponential Functions

- The domain of \( f(x) = b^x \) consists of all real numbers. The range of \( f(x) = b^x \) consists of all positive real numbers.
- The graphs of all exponential functions pass through the point \((0, 1)\) because \( f(0) = b^0 = 1 \).
- If \( b > 1 \), \( f(x) = b^x \) has a graph that goes up to the right and is an increasing function.
- If \( 0 < b < 1 \), \( f(x) = b^x \) has a graph that goes down to the right and is a decreasing function.
- \( f(x) = b^x \) is a one-to-one function and has an inverse that is a function.
- The graph of \( f(x) = b^x \) approaches but does not cross the \( x \)-axis. The \( x \)-axis is a horizontal asymptote.
# Transformations Involving Exponential Functions

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
</table>
| Horizontal translation          | \( g(x) = b^{x+c} \) | • Shifts the graph of \( f(x) = b^x \) to the left \( c \) units if \( c > 0 \).  
  • Shifts the graph of \( f(x) = b^x \) to the right \( c \) units if \( c < 0 \). |
| Vertical stretching or shrinking| \( g(x) = c \cdot b^x \) | Multiplying \( y \)-coordinates of \( f(x) = b^x \) by \( c \),  
  • Stretches the graph of \( f(x) = b^x \) if \( c > 1 \).  
  • Shrinks the graph of \( f(x) = b^x \) if \( 0 < c < 1 \). |
| Reflecting                      | \( g(x) = -b^x \)  
  \( g(x) = b^{-x} \) | • Reflects the graph of \( f(x) = b^x \) about the \( x \)-axis.  
  • Reflects the graph of \( f(x) = b^x \) about the \( y \)-axis. |
| Vertical translation            | \( g(x) = -b^x + c \) | • Shifts the graph of \( f(x) = b^x \) upward \( c \) units if \( c > 0 \).  
  • Shifts the graph of \( f(x) = b^x \) downward \( c \) units if \( c < 0 \). |
Use the graph of \( f(x) = 3^x \) to obtain the graph of \( g(x) = 3^{x+1} \).

**Solution**  Examine the table below. Note that the function \( g(x) = 3^{x+1} \) has the general form \( g(x) = b^{x+c} \), where \( c = 1 \). Because \( c > 0 \), we graph \( g(x) = 3^{x+1} \) by shifting the graph of \( f(x) = 3^x \) one unit to the left. We construct a table showing some of the coordinates for \( f \) and \( g \) to build their graphs.

<table>
<thead>
<tr>
<th>x</th>
<th>( f(x) = 3^x )</th>
<th>( g(x) = 3^{x+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( 3^{-2} = 1/9 )</td>
<td>( 3^{-2+1} = 3^{-1} = 1/3 )</td>
</tr>
<tr>
<td>-1</td>
<td>( 3^{-1} = 1/3 )</td>
<td>( 3^{-1+1} = 3^0 = 1 )</td>
</tr>
<tr>
<td>0</td>
<td>( 3^0 = 1 )</td>
<td>( 3^{0+1} = 3^1 = 3 )</td>
</tr>
<tr>
<td>1</td>
<td>( 3^1 = 3 )</td>
<td>( 3^{1+1} = 3^2 = 9 )</td>
</tr>
<tr>
<td>2</td>
<td>( 3^2 = 9 )</td>
<td>( 3^{2+1} = 3^3 = 27 )</td>
</tr>
</tbody>
</table>
The Natural Base $e$

An irrational number, symbolized by the letter $e$, appears as the base in many applied exponential functions. This irrational number is approximately equal to 2.72. More accurately,

The number $e$ is called the natural base. The function $f(x) = e^x$ is called the natural exponential function.
Formulas for Compound Interest

• After $t$ years, the balance, $A$, in an account with principal $P$ and annual interest rate $r$ (in decimal form) is given by the following formulas:

2. For $n$ compoundings per year:
   \[ A = P(1 + \frac{r}{n})^{nt} \]

3. For continuous compounding:
   \[ A = Pe^{rt} \]
Example

Use \( A = Pe^{rt} \) to solve the following problem: Find the accumulated value of an investment of $2000 for 8 years at an interest rate of 7% if the money is compounded continuously.

Solution:

\[
A = Pe^{rt} \\
A = 2000e^{(.07)(8)} \\
A = 2000 \ e^{(.56)} \\
A = 2000 \times 1.75 \\
A = $3500
\]
Exponential Functions