PERFECT PREIMAGES AND SMALL DIAGONAL

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Abstract. Hušek defines a space $X$ to have a small diagonal if each uncountable subset of $X^2$ disjoint from the diagonal has an uncountable subset whose closure is disjoint from the diagonal. It is known that the existence of a perfect preimage of $\omega_1$ which has a small diagonal is independent of the usual axioms of set-theory. In this note we prove that a perfect preimage of $\omega_1$ which is scattered will not have a small diagonal.

1. Introduction

We refer the reader to Gruenhage's interesting article [Gru02] for more background on spaces with small diagonal (see also [Zho82]). In particular, it is proven by Gruenhage that it is consistent with CH that each countably compact space with a small diagonal is metrizable, hence no countably compact preimage of $\omega_1$ could have a small diagonal. On the other hand, the authors prove in [DP06] that it follows from $\diamondsuit^+$ (a strengthening of CH) that there is a space with a small diagonal which maps perfectly onto $\omega_1$. In this paper we prove (in ZFC) that there is no scattered space with a small diagonal which maps perfectly onto $\omega_1$.

Hušek [Huš77], of course, originally asked about small diagonals for compact and $\omega_1$-compact spaces. The main open question in this area is whether every compact space with a small diagonal is metrizable. This statement has been shown to be consistent; for example it follows from each of CH and PFA. It is shown in [DP06, Proposition 18] that a counterexample will have to have a continuous image which does not have a small diagonal, hence it is interesting to consider preimages of those spaces that do not

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have a small diagonal. We offer the following problem as another interesting question about spaces with a small diagonal that may be easier to resolve in ZFC.

**Question 1.** If a compact space $X$ maps onto the Alexandroff double of the unit interval or of the compact double arrow space, will $X$ not have a small diagonal?

Using Hušek’s result that a compact non-metrizable space which has a small diagonal must have weight larger than $\omega_1$ and the result by Juhasz and Szentmiklossy [JS92] that it must have countable tightness the authors showed the following.

**Proposition 1.** [DP06, Corollary 5] If a compact space has a small diagonal, then it is metrizable if each of its separable subspaces is metrizable.

In fact, we should have stated the following strengthening because it uses the same proof.

**Proposition 2.** If a compact non-metrizable space has a small diagonal, then it has a countable discrete subset whose closure is not metrizable.

**Proof.** Assume that no countable discrete subset of $X$ is dense. Inductively select points $x_\alpha$ not in the closure of $D_\alpha = \{x_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$. Juhasz and Szentmiklossy [JS92] have shown that $X$ will have countable tightness (because it is compact and has a small diagonal). Therefore $Y = \bigcup_{\alpha < \omega_1} \overline{D_\alpha}$ will be compact and have a small diagonal. If each $\overline{D_\alpha}$ is metrizable, $Y$ will have net-weight, hence weight, equal to $\aleph_1$. By Hušek’s result $Y$ should be metrizable, which it is clearly not. □

It follows then that if a compact space with a small diagonal maps onto the Alexandroff double then the preimage of the non-isolated points will not be metrizable. In fact, more generally Gruenhage [Gru02, Corollary 2.5] has shown if a non-metrizable compact space with a small diagonal maps onto a metric space, one of the fibers will be non-metrizable.

2. **Perfect preimages of $\omega_1$**

It will be useful to recall the following reformulation of a space having a small diagonal.
Proposition 3. A space $X$ has a small diagonal iff for each uncountable family of pairs of points of $X$, $\{(x_\alpha, y_\alpha) : \alpha \in \omega_1\}$, there is an uncountable $A \subset \omega_1$ such that each point $x$ of $X$ has a neighborhood $U_x$ satisfying that $|U_x \cap \{x_\alpha, y_\alpha\}| \leq 1$ for all $\alpha \in A$.

The following is a simple generalization.

Lemma 4. If a space $X$ has a small diagonal and $\{F_\alpha : \alpha \in \omega_1\}$ is a family of finite subsets of $X$, then there is an uncountable $A \subset \omega_1$ such that each point $x \in X$ has a neighborhood $U_x$ satisfying that $U_x \cap F_\alpha$ has at most one element for each $\alpha \in A$.

Proof. Let $n \in \omega$ be chosen so that $A_0 = \{\alpha : |F_\alpha| = n\}$ is uncountable. For each $\alpha \in A_0$, let $\{F_\alpha(i) : i < n\}$ be an enumeration of $F_\alpha$ and let $\{P_j : j < \binom{n}{2}\}$ enumerate all the two element subsets of $n$. Recursively apply Proposition 3 to select uncountable sets $A_{j+1} \subset A_j$ so that each $x \in X$ has a neighborhood $U_x$ satisfying $|U_x \cap \{F_\alpha(i) : i \in P_j\}| \leq 1$ for each $\alpha \in A_{j+1}$. Clearly if $j = \binom{n}{2}$, then $A_j$ is the desired uncountable subset of $A_0$. \hfill \Box

It is nearly immediate now that no space with a small diagonal admits a finite-to-one perfect map onto $\omega_1$. We include this proof for the interest of the reader. Recall that a map is perfect if it is a closed map and the preimage of each point is compact.

Corollary 5. If $f : X \to \omega_1$ is a perfect surjective map and for some stationary set $S \subset \omega_1$, $|f^{-1}(\alpha)|$ is finite for each $\alpha \in S$, then $X$ does not have a small diagonal.

Proof. Let $S$ be a stationary set as in the statement of the Corollary. Since a countable union of non-stationary sets is again non-stationary, we may fix an integer $n$ so that $S_0 = \{\alpha \in S : |f^{-1}(\alpha)| = n\}$ is also stationary. For each $\alpha \in S_0$, choose a point $x_\alpha$ such that $f(x_\alpha) = \alpha + 1$ and let $F_\alpha = \{x_\alpha\} \cup f^{-1}(\alpha)$. Apply Lemma 4 to find an uncountable $A_0 \subset S_0$ such that each point $x \in X$ has a neighborhood $U_x$ satisfying $|U_x \cap F_\alpha| \leq 1$ for each $\alpha \in A_0$. Since $S_0$ is stationary, there is a $\lambda \in S_0$ that is a limit of $A_0$. By possibly shrinking the finitely many open sets, we can assume that $U_x \cap U_{x'}$ is empty for $x \neq x'$ with $f(x) = f(x') = \lambda$. Note that $F_\alpha \setminus \bigcup_{x \in f^{-1}(\lambda)} U_x$ is not empty for each $\alpha \in A_0$. It follows then that $A_0 \cap \lambda$ is contained in the image of the closed set $X \setminus \bigcup_{x \in f^{-1}(\lambda)} U_x$ while $\lambda$ is not. This implies that the map $f$ is not perfect. \hfill \Box
We will need to iterate the procedure from Lemma 4 in order to prove our main result. We adopt some notational convention to do so. Suppose we fix a sequence \( \{ x_\alpha : \alpha \in \omega_1 \} \) of points in a space \( X \). For any finite set \( F \subset \omega_1 \), let us use \( \hat{F} \) to denote the corresponding finite set \( \{ x_\alpha : \alpha \in F \} \). Similarly, for any uncountable collection \( \mathcal{F} \) of finite subsets of \( \omega_1 \), let \( \hat{\mathcal{F}} = \{ \hat{F} : F \in \mathcal{F} \} \), therefore be an uncountable collection of finite subsets of \( X \).

Next, for any uncountable set \( A \subset \omega_1 \) and integer \( n > 0 \), let \( \mathcal{F}_A^n \) denote the unique (canonical) partition of \( A \) into sets of size \( n \) such that \( \max F < \min F' \) (or conversely) for \( F \neq F' \in \mathcal{F}_A^n \). Finally, note that \( \bigcup F' \) (call it \( B \)) is an uncountable subset of \( A \) and that \( \mathcal{F}_B \) is a subfamily of \( \mathcal{F}_A^n \) because \( \mathcal{F}' = \mathcal{F}_B \).

As this notation builds up, it will be helpful to state the following simple fact.

**Lemma 6.** Let \( n, m \) be integers and let \( A \) be an uncountable subset of \( \omega_1 \). Let \( \mathcal{F}' \) be an uncountable subset of \( \mathcal{F}_A^n \) and let \( B = \bigcup \mathcal{F}' \). Then each member of \( \mathcal{F}_{n-m}^B \) is a union of \( m \) many pairwise disjoint members of \( \mathcal{F}' \).

We can now prove the main theorem.

**Theorem 7.** If \( X \) is a scattered space which maps perfectly onto \( \omega_1 \), then \( X \) does not have a small diagonal.

**Proof.** Assume that \( f \) is a perfect mapping from \( X \) onto \( \omega_1 \). Note that \( X \) is locally compact since, for each \( \lambda \in \omega_1 \), the set \( f^{-1}([0, \lambda]) \) is compact. For each \( \lambda \in \omega_1 \), we will let \( X_\lambda \) denote the points of \( X \) that map to \( \lambda \), and also note that \( X_\lambda \) is compact and scattered. Assume towards a contradiction that \( X \) has a small diagonal.

For each \( \alpha \in \omega_1 \), fix any point \( x_\alpha \in X \) such that \( f(x_\alpha) = \alpha \); thus we have chosen a fixed sequence of points \( \{ x_\alpha : \alpha \in \omega_1 \} \) as above. Recall that \( \mathbb{N} \) is the collection of all integer-valued functions with domain equal to some finite ordinal. We will inductively choose a collection, \( \{ A_t : t \in \mathbb{N} \} \), of uncountable subsets of \( \omega_1 \). In addition, we will also have selected \( \{ W_t : t \in \mathbb{N} \} \) consisting of covers of \( X \) by compact open sets. For each \( \emptyset \neq t \in \mathbb{N} \), let \( \pi(t) \) denote the usual integer product \( t(0) \cdot t(1) \cdot \ldots t(|t|-1) \), and let \( \pi(\emptyset) = 1 \).

To begin the induction let \( A_\emptyset \) denote the set \( \omega_1 \) and let \( W_\emptyset \) be any cover of \( X \) by compact open sets. Suppose that \( t \in \mathbb{N} \) is such
that $A_t$ has not been defined, but that (by induction) $A_{t'}$ and $W_t$ have been defined for all $t' < t$ in $\omega^* N$. Let $t' = t \upharpoonright (|t| - 1)$ be the immediate predecessor of $t$ and let $n$ denote the integer $\pi(t)$. We consider the family of finite sets $\mathcal{F} = \mathcal{F}^{A_{t'}}_n$ and the corresponding family $\hat{\mathcal{F}}$ of finite subsets of $X$. By Lemma 4, there is an open cover $W_t$ (consisting of compact open sets) and an uncountable subcollection $\mathcal{F}'$ of $\mathcal{F}$ such that $W \cap \hat{\mathcal{F}}$ has at most one element for all $W \in W_t$ and $F \in \mathcal{F}'$. By Lemma 6, it follows by induction that for $t' \subset t$, each member of $\mathcal{F}^{A_{t'}}_n$ is a union of $\pi(t)$ members of $\mathcal{F}^{A_{t'}}_{\pi(t)}$.

For each $t \in \omega^* N$, the set of accumulation in $\omega_1$ of the uncountable set $A_t$ will be a cub in $\omega_1$. Since the intersection of countably many cub’s of $\omega_1$ is again a cub, we may choose a limit $\lambda \in \omega_1$ such that $A_t \cap \lambda$ is cofinal in $\lambda$ for each $t \in \omega^* N$. Observe then that for each $F \in \mathcal{F}^{A_{t'}}_n$, with $\min F \in \lambda$ we also have $F \subset \lambda$ since $\lambda \cap (A_t \setminus \min F)$ is infinite and $\max F < \min F'$ for all $F' \in \mathcal{F}^{A_{t'}}_n$ such that $F' \setminus \min F$ is not empty.

Now we begin to inductively choose a finite sequence $t$ of integers (hence $t \in \omega^* N$) and a descending sequence of ordinals (which must therefore stop in finitely many steps). Let $\gamma_0$ denote the maximum non-empty scattering level of $X_\lambda$ (which must exist since $X_\lambda$ is compact and non-empty). Set $t(0)$ to be any integer greater than the finitely many points of $X_\lambda$ at scattering level $\gamma_0$. If we have defined the first $k$ elements of $t$ we will use $t \upharpoonright k$ to denote that function, even though we don’t yet know what $t$ is. Let $W_0 \subset W_t$ (with $|W_0| < t(0)$) be a cover of those fewer than $t(0)$ many points at scattering level $\gamma_0$ of $X_\lambda$. Set $U_0 = \bigcup W_0$ and note that $\hat{F} \setminus U_0$ is not empty for each $F \in \mathcal{F}^{A_{t(0)}}$. 

Assume now that we have defined $t(i), \gamma_i$ and $W_i$ for $i < k$ such that $|W_i| < t(i)$, $W_i \subset W_{t(i+1)}$, and $X_\lambda \setminus \bigcup_{i < k} W_i$ has scattering height less than $\gamma_{k-1}$. We continue as follows. Set $U = \bigcup_{i < k} W_i$; if $X_\lambda \setminus U$ is empty we stop. Otherwise, let $\gamma_k$ be the maximum non-empty scattering level of $X_\lambda \setminus U$ and let $t(k)$ be any integer larger than the cardinality of that level. Choose
\(W_k \subset W_{t|k+1}\) to be any fewer than \(t(k)\) many sets which covers that finite set of points of \(X_\lambda \setminus U\) at scattering level \(\gamma_k\).

Recall from above that we noted that for each \(F \in \mathcal{F}_{t(0)}^{A;_t}\), \(\widehat{F} \setminus \bigcup W_0\) is not empty. By Lemma 6 each \(F \in \mathcal{F}_{\pi(t|2)}^{A;_t}\) is a union of \(t(1)\) many pairwise disjoint members of \(\mathcal{F}_{t(0)}^{A;_t}\). Therefore, since \(|W_1| < t(1)\), it follows that \(\widehat{F} \setminus (\bigcup W_0 \cup \bigcup W_1)\) is not empty for each \(F \in \mathcal{F}_{\pi(t|2)}^{A;_t}\). By a straightforward induction it then follows that for each \(F \in \mathcal{F}_{\pi(t)}\) we have that \(\widehat{F} \setminus \bigcup \{\bigcup W_i : i < |t|\}\) is not empty.

We are now ready for our contradiction. Choose any sequence \(\{F_n : n \in \omega\} \subset \mathcal{F}_{\pi(t)}^{A;_t}\) such that \(\{\min F_n : n \in \omega\}\) is cofinal in \(\lambda\). Recall also that max \(F_n \in \lambda\) for each \(n \in \omega\) as well. For each \(n\), choose \(y_n \in \widehat{F} \setminus \bigcup \{\bigcup W_i : i < |t|\}\). It follows now that \(\{f(y_n) : n \in \omega\}\) is a closed subset of \(X \setminus \bigcup \{\bigcup W_i : i < |t|\}\) since \(X_\lambda\) is contained in \(\bigcup \{\bigcup W_i : i < |t|\}\). \(\square\)

**Question 2.** If a space \(X\) has a small diagonal and maps perfectly onto a space \(Y\) with point preimages being scattered, will \(Y\) also have a small diagonal?

The formulation and proof of Theorem 7 can easily be strengthened to only require that the map be a closed map onto a stationary subset of \(\omega_1\), and that point preimages are locally compact scattered rather than the whole space is scattered. In addition, it is easy to check that a compact scattered space with a small diagonal is countable and metrizable. Combining these ideas yields the following result.

**Proposition 8.** Suppose that a space \(X\) maps onto a subset \(S\) of \(\omega_1\) by a closed mapping such that fibers are compact and scattered. Then \(X\) has a small diagonal if and only if \(S\) is not stationary and the point preimages are also countable.

**References**


