December 11, 2002

Your name ________________________________

It is important that you **show your work**. The total value of this test is 190 points.

1. (20 points) Find a relation \( R \) on the set \( \{1, 2, 3, 4\} \) satisfying each set of requirements. Leave your answer in matrix form.
   
   (a) Reflexive, antisymmetric and symmetric.
   
   (b) Not symmetric and not antisymmetric.

2. (20 points) Prove that

\[
1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}
\]

for \( n = 0, 1, 2, \ldots \).
3. (15 points) Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 217 and 975. In other words, find integers $x$ and $y$ such that $gcd(217, 975) = 217x + 975y$.

4. (10 points) Find the remainders when each $N$ is divided by $d$.

   (a) $N = 5^{41}$ and $d = 7$

   (b) $N = 123,456,789,101,112$ and $d = 11$
5. (20 points) Find the base 4 and base $-4$ representation of each of the numbers below.

(a) 217
(b) 36.75

6. (15 points) Suppose $A$, $B$, and $C$ are three sets that satisfy $|ABC| = |AB\bar{C}| = |A\bar{B}C| = |A\bar{B}C| = |\bar{A}BC| = |\bar{A}BC| = |\bar{A}BC|$ and $|A| = 20$, $|B| = 17$, and $|C| = 19$. What is $|A \cup B \cup C|$? Recall that $UV$ refers to $U \cap V$.

7. (20 points)

(a) Prove that the intersection of two symmetric relations on the set $A$ is also symmetric.

(b) Prove that every subset of an antisymmetric relation is antisymmetric.
8. (20 points) How many relations $R$ on the set $S = \{a, b, c\}$ are there such that

(a) $R$ is not reflexive?
(b) $R$ is antisymmetric?
(c) $R$ is an equivalence relation.
(d) $R$ is reflexive, antisymmetric and transitive. (very hard!)
9. (20 points) Let \( S = \{1, 2, 3, 4, 5, 6\} \).

(a) How many four-digit numbers can be made using the members of \( S \) as digits if repetition of digits is allowed.

(b) How many four-digit numbers can be made using the members of \( S \) as digits if repetition of digits is not allowed.

(c) How many four-digit multiples of 15 can be made using the members of \( S \) as digits if repetition of digits is allowed.

(d) How many four-digit multiples of 6 can be made using the members of \( S \) as digits if repetition of digits is not allowed.
10. (15 points) Let $S = \{a, b\}$ and let $R = \{(a, a), (a, b), (b, a)\}$.
   
   (a) Prove that $R$ is not antisymmetric.
   (b) Prove that $R$ is symmetric.
   (c) Prove that $R$ is not transitive.

11. (15 points) Let $Z$ denote the set of all integers. Define the relation $R$ on $Z$ by $xRy$ if $x - y$ is even.
   
   (a) Prove that $R$ is reflexive.
   (b) Prove that $R$ is symmetric.
   (c) Prove that $R$ is not antisymmetric.
   (d) Prove that $R$ is transitive.
   (e) Find the cells $[0]$ and $[1]$. 